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Communication Network Design and Evaluation Using Shadow Prices.

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To my Parents Alicia and César

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Abstract

We use shadow prices to measure the rate of change of performance functions of a communications network with respect to incremental changes in one or more of the design parameters such as capacity, anticipated demand, reservation levels, routing proportions, etc. The ability to evaluate such shadow prices allows us to introduce designs that optimize network performance according to network-wide criteria. We present models for the evaluation of shadow prices for different cases of networks: single-rate circuit-switched networks, wireless networks, multi-rate circuit-switched networks. For these cases, we demonstrate the wide applicability of shadow price methods. We formulate the new and very useful notion of sum capacity that can be used to compare different adaptive routing schemes, we can determine the actual pricing of calls to reflect their effect on the entire network, we trade off new call blocking with the forced termination of calls due to handoff drops in wireless networks, we dimension networks to achieve a prescribed set of new call and handoff blocking probabilities, we determine the tradeoffs between calls of different rates in multi-rate networks. We also point out problems of current interest which can be addressed by the shadow price framework and outline a brief methodology.

Chapter 1

Introduction

The area of communications has generated a considerable interest among scientists and practitioners in the last decade. Many issues have been investigated such as network routing, mathematical modelling and network optimization. Moreover, technological developments have influenced the communication networks which have evolved to form different classes such as circuit-switched, integrated and wireless networks. The design of such networks is a process which in part is general and it does not depend of the kind of network to be used. One of the most important steps in the design is the analysis of the network. Communication networks must be analyzed with respect to different criteria. The criteria selection is also an important part of the analysis and basically is concentrated in capacity of resources, cost and performance. Performance includes requirements as those on response times, delays and the total amount of traffic handled in a period of time.

Analytical modelling is the most common technique for network system analysis, where mathematical models that are feasible for implementation give performance measures with relative ease. An alternative to analytical modelling is simulation of the statistical behavior of the network. Performance measures given by the analytical models set up the fundamentals for decision making in the design and analysis process. Such measures are basically given in terms of quality of service, end-to-end blocking, average utilization, transmission costs, delays, carried traffic, etc.

Assuming that the network to be designed has a topology which is already defined by a connectivity matrix, the components such as transmission and switching systems must be sized subject to grade of service and other performance measure constraints. This stage in the design process is part of the network management, where two very important functions take place: Routing and Dimensioning.

Routing solves the problem of allocating the traffic so that the quality of service remains within acceptable levels considering that a topology of the network, the capacities of the resources in the network and the external demands are known. Dimensioning solves the problem of assigning capacities to the resources such that for the present demand the quality of service is within acceptable levels considering that the topology of the network and the external demand are known.

There are different kinds of routing schemes, from centralized and distributed to fixed and adaptive, some of them are briefly presented in Chapter 2.

Adaptive routing is about finding the *best* path in the network such that connecting a call through it will cause the least damage to future customers, i.e., find spare capacity in the network and based on some decision rules choose that path that minimizes blocking. Adaptive routing schemes offer much improvement in terms of performance. They improve network blocking, provide ability to withstand mismatch between the network capacity and the level of offered traffic and provide robustness to changing traffic.

Some of the advantages of adaptive routing are the survivability and self-healing properties of the network in the presence of failures. Recently, Markov decision algorithms have been introduced to analyze routing performance as in [42], [43], [26], [22] and [23], which facilitates the dimensioning and optimization of networks. Much of the literature on the subject of adaptive routing schemes consists of discussions of the feasibility of implementation and the development of reasonably accurate performance evaluation algorithms to compute network blocking, but network blocking does not characterize entirely the benefits of adaptive routing. Features such as robustness to failures, complexity of switching nodes, gathering of information such as state occupancy, capacity mismatch, traffic optimization and survivability of the network are equally important and need a comprehensive study to reflect the overall benefits of these schemes.

As a result, **the main goal** of this research has been to provide an analytic framework based on shadow prices that characterizes performance such that it can be applied for routing and dimensioning of networks. One of the applications of this analytic tool is to compare performance of adaptive routing schemes in single and multi-rate networks and reservation schemes in wireless networks. We have concentrated the study to capacity mismatch and traffic optimization. Performance evaluation of some routing schemes and maximization of carried traffic have been done for circuit-switched networks and are extended to the case of multiple classes of customers and wireless networks, eventhough the techniques can be applied to different networks such as Asynchronous Transfer Mode (ATM) networks. This research contains an analysis of routing policies as well as practical engineering aspects examining the trade-offs between the aforementioned characteristics in the network. In the following section the motivation and objectives of this research are explained, and in Section 1.2 the contributions of the research are highligthed to end with a brief explanation of the contents of the chapters in this work.

1.1 Motivation

The benefits of adaptive routing schemes have not been quantified because of the need of a thorough understanding of their behavior. Some examples have been presented in the literature which consider assumptions that simplify the analysis, e.g., independence of the birth-death processes that determine the states of the links in the network. Others have compared in terms of network blocking or Origin-Destination (OD) blocking probabilities adaptive routing schemes in the same network arriving to conclusions of improvement of one routing scheme over others, but this is not a fair comparison of adaptive routing schemes since the conclusions are made depending on the external arrival rates and capacities on the links as well as the topology of the network. It would be possible to have a network where for light traffic load one routing scheme performs better than the other, whereas for heavy traffic load the opposite happens.

The need of a unified theory to compare performance of routing schemes and their behavior is one of the main motives of this research. The need to quantify the benefits of adaptive routing through performance measures such as network blocking and network rate of return is part of the framework presented in this work. Network management functions such as admission control and routing, have become more complex as technology advances, and decision criteria for routing on a per call basis is needed. Distributed decision making is important to achieve high efficiency in networks, and having available the network rate of return can help to improve the process, but several variables play an important role and affect performance, then a sensitivity analysis of such performance measures can determine a set of decision rules that can be used for routing. The use of shadow prices of the network rate of return as a tool for network management is also a motivation for this research. The extension of the methodology to different networks is evident since the shadow prices can help improve performance. Due to the increasing demand on Personal Communication Systems (PCS), the applications on wireless networks are important, we introduce some examples of shadow prices on these networks.

1.2 Contributions

This research has consisted of the analysis of performance evaluation through OD pair blocking probability of several adaptive routing schemes with models as those described in Chapter 3 for asymmetric and symmetric fully connected networks, the calculation of shadow prices of the network rate of return, the formulation of a nonlinear constrained maximization problem solved by using the shadow prices and the comparison of several routing schemes, in terms of their sum capacity, results based on those concepts are extended to circuit switched multirate networks using an adaptive routing scheme, such as LLR. A fixed point algorithm for K classes of traffic is presented. Examples of the performance of LLR for networks with two classes of traffic and with trunk reservation levels are shown. The methodology followed to compute the shadow prices is also applied to the fixed point algorithm of a wireless network model with reservation of channels for handoffs.

1.2.1 Shadow Prices and Sum capacity

The first contribution of this research is the unified theory to compare different adaptive routing schemes by their sum capacity. The concept of *Sum Capacity* is defined as the largest sum of exogenous arrival rates that the network topology can accomodate while maintaining a prescribed blocking probability for every OD pair.

The comparison of LLR and ALBA has been carried out for small networks. This comparison has been presented in a conference [70] where ALBA uses randomization to brake ties on routes, and is developed in full detail also in [67] where ALBA uses a sequence to brake ties. LLR has been considered with and without trunk reservation, and for ALBA the number of aggregate states has been varied. It has been seen that increasing the number of aggregate states in ALBA makes its performance close to that of LLR. The evaluation of the network rate of return and its comparison between the routing schemes is also presented. The rate of return represents the revenue obtained by the carried traffic and since this function depends on the OD pair blocking probabilities it is also related to the performance evaluation algorithm. From these comparisons it was seen that the performance of LLR without trunk reservation was better than any other of the cases analyzed. Also in the networks used, ALBA performs better when the number of aggregate states increases as can be seen in the figures of Chapter 3 where it is shown that for the cases of three and four nodes LLR with trunk reservation has better performance but for a 5-node network ALBA becomes better than LLR with trunk reservation.

The complexity of the performance evaluation algorithm of LLR as well as the shadow prices calculation is presented in Section 4.6. The shadow prices of the rate of return with respect to exogenous arrival rates are calculated and compared in Chapter 4. The use of shadow prices to maximize the carried traffic in the network can be found in Chapter 5, in which a gradient descent algorithm is used. The maximization is performed under a maximum OD pair blocking probability constraint. The algorithm stops when all the OD pairs in the network have reached a maximum blocking probability and the result is the maximum exogenous traffic necessary to give that blocking in a network under the routing scheme. It has been seen that as in the performance evaluation analysis, LLR without trunk reservation performs better than the other cases analyzed, and that ALBA with increasing number of aggregate states approaches the performance of LLR. The comparison of ALBA using randomization and sequencing using sum capacity is presented.

1.2.2 Bounds

It is not clear how routing schemes should be optimized because there is strict no definition of an optimal routing policy, hence a point of comparison is needed that could tell how well or bad a scheme performs. This point of comparison is provided by bounds on the performance measures such as the Erlang bound or the Max-Flow bound in [18]. In this research, we use those bounds defined in [18] and [17] and explained in Section 3.6 to compare optimized and non-optimized performance measures for LLR and ALBA. The bound that is closest to the results obtained is presented for the network rate of return and the sum capacity.

1.2.3 Modified LLR (MLLR)

A modification of LLR has also been analyzed and is presented in full detail in Section 3.5, it is called Modified LLR or MLLR. This routing scheme considers a distribution of the external arrival rates which determines how traffic will be offered to all available routes for each OD pair. This distribution consists of proportions of traffic called load sharing coefficients. LLR is the case in which the distribution indicates that all traffic is offered to the direct single-link route. The shadow prices of the rate of return are calculated with respect to the load sharing coefficients for a fixed external arrival rate. The optimization by the gradient descent algorithm gives the load sharing coefficients that maximize the rate of return for a given external arrival rate, since one of the choices of the load sharing coefficients gives LLR, then with the optimization, the final result should perform at least as good as LLR. It has been obtained that for small networks that are highly asymmetric there are load sharing coefficients different from those that use direct link preference which allow MLLR to perform better. Even though in both cases analyzed this difference is small, it is possible to obtain a network using MLLR that gives better performance than that of LLR.

1.2.4 Multi-Rate Networks

Adaptive routing schemes are anticipated to be more effective in handling multiple classes of calls such as those arising from voice, data or other wideband services.

Real Time Network Routing (RTNR) [5] is one routing scheme that takes into account multiple classes of services.

Little work has been done to date on the effect of multiple call types on the performance of adaptive schemes. With some modifications, adaptive routing schemes may be suitable for routing and congestion control in this kind of networks.

The performance analysis of a network supporting K classes of customers using LLR is presented in Chapter 6 where the calculation of the shadow prices is obtained with respect to the arrival rates of the classes, and optimization of the scheme in terms of the sum capacity is calculated for small networks. It is shown by comparison of numerical evaluation and simulation that trunk reservation levels play an important role in the OD pair blocking probability with the fixed point algorithm presented, making the approximation more accurate and decreasing blocking.

1.2.5 Application to Wireless Networks

Finally, the methodologies used can be extended to compare channel allocation schemes, reservation policies, acceptance of new calls vs. handoff calls, etc., in wireless networks. A fixed point model which considers mobility, capacity and new call arrival rates as control variables is presented and its performance compared to simulation results, the shadow prices are used to obtain sensitivity analysis of carried traffic and minimization of blocking probability calculating handoff rates and giving priority by channel reservation. This is done by using a revenue function that considers a cost for every handoff call that fails to be connected. We use Fixed Channel Assignment (FCA) with priority for handoffs over new call arrivals by reserving a number of channels in all the cells. The performance measures used are new call blocking and drop handoff probabilities. The shadow price is calculated for a performance function called the network net revenue which considers the revenue generated by accepting a new call arrival into the network as well as the cost of rejecting a handoff attempt in any cell.

The shadow prices are shown to be very useful in determining subscriber pricing especially in the context of non-uniform traffic and unequal mobilities. We calculate shadow prices of the net revenue with respect to the new call arrival of each cell and introduce a framework for pricing of call services. Simulation and numerical results are presented as well as confidence intervals showing the accuracy of the model. This suggests a method of differential pricing for the different network services expected to be a part of PCS which is related to the network consequences of providing that service. Finally, as an application of shadow prices, we formulate a nonlinear constrained optimization problem to calculate the sum capacity for a given network by maximizing the net revenue using shadow prices in a gradient descent algorithm. Comparison of the sum capacities indicate that the optimization using shadow prices results in a significant improvement. This provides evidence that matching capacity distribution to traffic is important in wireless networks.

1.3 Organization

The organization of the dissertation is as follows. In Chapter 2, the technical background on routing is described, as well as its evolution, and briefly some important results and techniques used for performance evaluation as the Erlang Fixed

Point Approximation and the implied cost methodology are explained. In Chapter 3, the notation is introduced and the fixed point equations for LLR and ALBA are presented. In the same chapter a modification of LLR which will be called MLLR is explained. The performance of all routing schemes has to be compared with bounds on blocking probability and traffic carried, since it is not clear what an optimal routing scheme is. These bounds are explained in Chapter 3 and used in Chapter 4 and Chapter 5 for the network blocking and network sum capacity, respectively. The calculation of the shadow prices for LLR, ALBA and MLLR is described in Chapter 4. An algorithm to obtain the shadow prices along with numerical results of the calculation are also presented in Chapter 4. In Chapter 5 we formulate the optimization problem used to calculate the sum capacity subject to a blocking probability constraint and report the numerical results for LLR, ALBA and MLLR. In Chapter 6, the performance evaluation, as well as the shadow price calculation for multi-rate networks using LLR is presented. Chapter 7 contains an application of the concepts of shadow prices to channel reservation levels for handoff arrivals in cellular networks and maximization of net revenue by using the shadow prices in a nonlinear optimization problem. Finally, in Chapter 8 the conclusions are presented.

Chapter 2

Routing in Networks

Network topology, network management methods and service demand are factors that affect performance in a network. Routing is a network management method and is formed by a set of decision rules that determine the connection of customers on a per call basis. This set of rules specifies, for each new arriving customer in the network, the path through which the connection will be set up and the data required to make that decision such that the quality of service is not degraded more than the levels allowed. In this chapter a brief review of the evolution of routing schemes is presented, as well as some of the results obtained that are of importance to the research of this work.

2.1 Routing Evolution and Classification

Currently, a significant majority of the routing policies in circuit switched networks are Fixed Alternate Routing (FAR) schemes [4], [20]. However, the widespread incorporation of intelligent digital switches (e.g., 5ESS or DMS100) and digital transmission combined with common channel signaling has made possible more sophisticated routing schemes which are real-time adaptive and based on network state information. Examples are Dynamically Controlled Routing (DCR) [72], Real-Time Network Routing (RTNR) [5], Residual Capacity Adaptive Routing (RCAR)

[25], Least Loaded Routing (LLR) [8], Aggregated Least Busy Alternative Routing (ALBA) [54], State-Dependent Routing (SDR) [58] [40] and Forward Looking Routing (FLR) [39].

Network routing evolution has been given by technological advances, improvements in quality of service and introduction of new services. The improvement in performance is achieved by adaptivity and robustness to traffic changes and network failures. A way of providing the flexibility to adapt to changing traffic demands and cope with traffic patterns and new services is through the use of Dynamic Routing, which in general is a routing scheme where the decision rules vary over time. Adaptive routing also provides the flexibility needed to compensate traffic changes and network failures. A routing scheme is adaptive when its decision rules change based on network state at the time the decision must be made. Thus, routing schemes can be classified in a broad sense as dynamic and adaptive depending on the form the decision rules change. In the following an overview of different routing schemes is introduced, organized according to their evolution and/or classification. Most of this information can be found in several sources, one of them is [20].

2.1.1 Fixed Hierarchical Routing (FHR)

This routing is the oldest and most used scheme. The switches for FHR are organized in a hierarchy where the ones at the bottom of the hierarchy are origin/destinations, and switches at higher levels serve as transit switches. The route selection is static where the trunk groups associated with a particular call are ordered in a fixed sequence, where the sequence is determined by the hierarchy.

The routing policy consists of fixed alternate routes with restrictions on the paths that can be used imposed by the hierarchy of the offices which in the North American network is of five classes or groups.

An arriving call will be routed through the first available group in a sequence of alternate routes. This sequence is defined by the OD pair, the rule is to attempt the group which is closest, i.e., with lowest hierarchy to the destination, provided the group exists. The call will be lost if there is no available path.

Some of the advantages of this routing scheme are that it does not introduce cycles in the routing, it can be implemented with simple hardware.

2.1.2 Dynamic Nonhierarchical Routing (DNHR)

This routing scheme was introduced in the mid 1980s in the long distance AT&T network [4]. The hierarchy relation between offices is no longer considered, hence all switches can receive requests for connections as well as serve as *intermediate* nodes in the multi-link paths.

The routing considers two-link alternate routes, and it is of the dynamic type in the sense that the set of routes for each origin-destination changes according to traffic matrices computed off-line and that represent the demand for several periods of the day. Its routing decisions change at fixed time intervals. It does not use any network information feedback.

2.1.3 Alternate Routing Schemes

Alternate routing schemes provide, for each arriving call, the possibility of selecting a path from a set of available routes which includes the single-link and alternate routes. This set is ordered in a sequence and in case an alternate route is needed to connect the call, the first path that can carry it is chosen to make the connection. DNHR uses alternate routing in a dynamic fashion by table look-up, another one of these routing schemes is Fixed Alternate Routing.

2.1.3.1 Fixed Alternate Routing (FAR)

This is also one of the oldest and simplest routing schemes. Each origin-destination pair has in advance a set of available paths to connect the arriving calls, which is ordered in a sequence and it is generally smaller than the complete set of routes between origin and destination.

The policy is to attempt to connect the call in the single-link route, if there is capacity available, then the call will be set up. If there is no free capacity, then the call overflows to the set of alternate routes and is attempted on these paths in the ordered given by the sequence. The call will be connected on the first path with free capacity found in the sequence, if no such path exists or its capacity is fully occupied, then the call will be blocked and lost.

2.1.4 Adaptive Routing Schemes

Adaptive routing schemes give for each arriving call the possibility of selecting a path from a set of available routes according to a value placed on each path depending on observations of some of the network components. Since these observations are based mostly on the state of the network defined by the occupancies of the links, these schemes are also referred as State Dependent Routing schemes. These schemes adapt to changing conditions.

In this section some of the adaptive routing schemes and their policies are mentioned. The importance of adaptive routing schemes in telecommunication networks is because of their improvement of performance measures such as network blocking and carried traffic as well as robustness to network failures. Their disadvantage is in the complexity of the algorithms used to determine this measures because the need of state information from the network.

2.1.4.1 Residual Capacity Routing (RCAR)

This routing scheme uses occupancy information from all the links in the network updating it periodically by measurements made by the switches.

The first proposal to implement an adaptive routing scheme of the residual capacity type was called Dynamic Call Routing (DCR) [7]. This routing sends the calls through those paths with the largest expected number of free circuits. Each OD pair has a single-link route and one alternate route available which is computed periodically. Every incoming call is attempted first in the single-link route. Calls which are blocked on the single-link path are offered to the alternate path, and in case of no free capacity the call is blocked. Alternate paths are selected at random with a distribution proportional to the estimated residual capacity on all possible paths for each OD pair. DCR considers a centralized processor which updates the information needed to make the decisions and sends it to the switches every ten seconds. The evolution of routing has permitted to make the decisions in a distributed manner, and the first extension of DCR was to eliminate the dependance on a central processor. Dynamic Alternative Routing (DAR) does that.

2.1.4.2 Dynamic Alternative Routing (DAR)

DAR [18] is a simple decentralized routing scheme that uses local information to make the decisions. Although intended for a fully connected network, DAR can be extended and applied to different topologies [37]. This routing scheme also uses occupancy information from all the links. Alternate routes are limited to two-link paths, trunk reservation is introduced.

The policy for this scheme is as follows. An arriving call is offered to the single-link route, and it is connected if there is free capacity on that link. If the single-link path is busy, then it is offered to a two-link alternate route which is stored in the switch. In case of capacity available satisfying the trunk reservation parameter, the call will be connected and the two-link path will still be stored, otherwise the call will be blocked and a new two-link path will be chosen randomly.

For a performance evaluation algorithm and bounds on the performance see [18] where the analysis of DAR is done based on the bounds which are also used in this dissertation in Section 3.6. Shadow prices have also been calculated for DAR in [38] where they are used for network dimensioning (assign capacities to links).

2.1.4.3 State Dependent Routing (SDR)

The simplest form of state dependent routing is to route the incoming call through links that have the largest free capacity. SDR attempts to measure the cost of accepting a call on a route. This routing scheme was proposed in [40], and in [42] it can be seen the application of Markov decision algorithms for SDR. SDR is a *cost-based routing* scheme with the cost function reflecting the expected number of future lost calls on certain link, which can be thought of as a state-dependent implied cost. The set of paths available consists of the single-link route and the two-link alternate routes. An arriving call will be connected through that path with minimum cost as long it is less than one. For performance evaluation and further modifications of this routing policy see [27], and [43].

2.1.4.4 Real Time Network Routing (RTNR)

The routing policy for this scheme was introduced in [5]. A performance evaluation algorithm for RTNR with one class of traffic is in [6]. Single rate RTNR is an implementation of ALBA as their behavior is based on the use of groups of states to determine occupancy levels.

This routing scheme considers the states of the links grouped in six aggregate states ranging from least loaded to reserved states. This states are updated in an adaptively form every three minutes by changes in traffic demand. This scheme is applied to multirate networks with different Grade of Service (GoS) for each class.

An arriving call is attempted in the direct link first and if there is free capacity then it will be connected, otherwise the originating switch requests a bit map for each of the six aggregate states and each of the links incident to the destination switch. The bit map for a specific aggregate state gives the links that are at that occupancy level. With this information, the originating switch determines the state of the two-link paths that connect origin and destination and chooses the least loaded path. The routing policy also considers trunk reservation which is also computed adaptively depending on the traffic demand for each OD pair.

2.2 Methodologies for Performance Analysis

There are many different performance evaluation algorithms, most of which are based on Markov Processes to describe qualitatively the schemes. Recently the techniques of Markov Decision Processes have been applied to routing in circuit-switched networks ([39], [40], [58], [42]), and broadband networks in [45] and [46]. While others have applied learning automata as in [55], [56] and [57].

In this section the fundamental model for performance evaluation, the Erlang Fixed Point approximation, is explained, as well as the introduction of shadow prices to the analysis of routing schemes. The section ends with some results to be mentioned of related work.

2.2.1 Erlang Fixed Point Approximation

This method is the most used to compute the blocking probability of a link in telecommunication networks. The problem to which the approximation was applied first can be considered as follows.

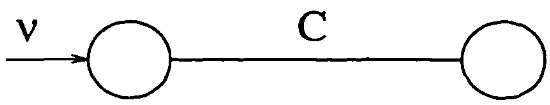

Figure 2.1: Link with Capacity C offered Poisson Traffic with mean ν

Assume a link with finite capacity C is offered Poisson traffic with arrival rate ν, as in Figure 2.1. A call will be blocked and lost if all C circuits are busy, otherwise, the call is accepted and occupies one circuit for its holding time which is exponentially distributed with mean $1/\mu$ and independent of other holding times and earlier arrivals. Then Erlang's formula

$$E(\nu, C) = \frac{(\nu/\mu)^C}{C!}\left[\sum_{n=0}^{C} \frac{(\nu/\mu)^n}{n!}\right]^{-1}, \tag{2.1}$$

gives the proportion of calls that are lost in the link, i.e., it gives the stationary probability that all C circuits are busy.

2.2.1.1 Network using Fixed Routing

In [31] a network using fixed routing was analyzed and a closed product form describing its state was obtained. We explain briefly that network in this section. For the reduced load approximation that determines the offered loads to the links of a network using fixed routing see [74].

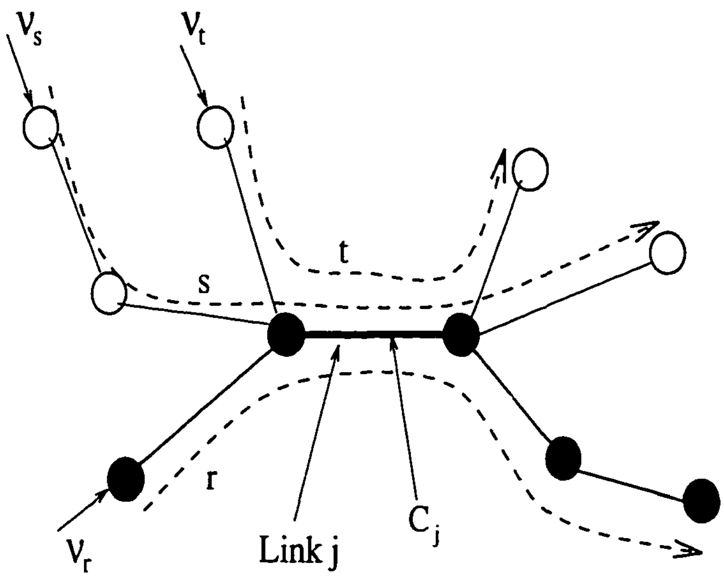

Figure 2.2: Network Using Fixed Routing

Consider a network where the links are numbered $1, 2, \ldots, J$, with the capacity of link j being C_j let $\mathcal{R}$ be the set of routes in the network. Figure 2.2 explains the notation for a network and some routes with fixed routing. A route $r \in \mathcal{R}$ is a path formed by a set of links joining origin and destinations. Calls arrive as a Poisson process with mean ν_r requesting service on route $r \in \mathcal{R}$. A call on route r is blocked and lost if on any link of the route there is no free circuit to connect the call.

Assume that call holding times are exponentially distributed with unit mean. Define the state of the network to be $\mathbf{n} = (n_r, \ \forall\ r \in \mathcal{R})$ where n_r is the number of calls present on route r. Then the stationary distribution for this process has a product form [32]

$$\pi(\mathbf{n}) = \prod_{r \in \mathcal{R}} \frac{\nu_r{}^{n_r}}{n_r!} \left[\sum_{n \in \Omega} \prod_{r \in \mathcal{R}} \frac{\nu_r{}^{n_r}}{n_r!} \right]^{-1}, \quad n \in \Omega, \tag{2.2}$$

where Ω is the set of states such that the number of calls on any link does not exceed its capacity. A quantity of interest to obtain is the blocking probability on any route $r \in \mathcal{R}$, L_r. The Erlang fixed point approximation gives that blocking probability, L_r, on route r as follows

$$1 - L_r = \prod_{j \in r} [1 - E(\rho_j, C_j)], \tag{2.3}$$

where ρ_j is the total offered traffic to link j and $E(\rho_j, C_j)$ is the Erlang formula given by equation (2.1). The offered traffic to link j is given by a reduced load approximation where the contribution from all the routes that use link j is considered as follows

$$\rho_j = \sum_{r: j \in r} \nu_r \prod_{i \in r - \{j\}} (1 - B_i), \tag{2.4}$$

where $B_i = E(\rho_i, C_i)$ is the blocking probability of link i.

It can be seen from Figure 2.2 that it contains three fixed routes r, s and t with external arrival rates ν_r, ν_s and ν_t, respectively, and all these routes use link j with capacity C_j. The offered traffic to link j is be given by equation (2.4), where the summation is over the three routes r, s and t.

2.2.2 Implied Cost Methodology

Implied costs are measures of the rate of change of an objective function with respect to the parameters of the network. The implied cost methodology was used in [35], [36] and [29] to calculate shadow prices of the network rate of return using fixed routing, where the rate of return was the sum over all the routes in the network of the carried traffic weighed by the revenue of accepting a call on each route. When these revenue coefficients are identically one, the rate of return gives the total traffic carried in the network. Let w_r be the revenue generated by accepting a call on route $r \in \mathcal{R}$, let λ_r be the external arrival rate for route r, let L_r be the blocking probability of route r, and define the network rate of return W as

$$W(\underline{\lambda}, \mathbf{C}) = \sum_{r \in \mathcal{R}} w_r \lambda_r (1 - L_r), \tag{2.5}$$

where the notation reflects the dependance on all the exogenous arrival rates $\underline{\lambda}$ and the capacities of all the links in the network $\mathbf{C}$.

The approach in [35] associates an *implied cost* with each link, c_k, which measures the knock-on effects on the entire network of carrying an additional call on that link, giving that the true cost of carrying a call on a route is the surplus value, i.e., the difference of what we earn, w_r, and what it costs, $\sum_{k \in r} c_k$, to route the call on path $r \in \mathcal{R}$. The implied costs c_k are given by the fixed point equation

$$c_k = \eta_k (1 - B_k)^{-1} \sum_{r : k \in r} \lambda_r \left(c_k + w_r - \sum_{j \in r} c_j \right), \tag{2.6}$$

where $\eta_k = E(\rho_k, C_k - 1) - E(\rho_k, C_k)$, and B_k is the blocking probability of link k.

2.2.3 Shadow Prices

Shadow prices were introduced in [35] for a network with fixed routing and extended to alternate routing. In [36] the methodology was extended to the case of networks with trunk reservation parameter on the links. Shadow prices are used to determine how calls should be routed or allocated in a network so as to optimize performance.

Shadow price techniques are based on calculating the derivatives of an implicitly defined function and expressing them as functions of certain variables, i.e., application of the chain rule to those functions. The variables such as the exogenous arrival rates and capacities of the links, and the derivatives in question form a set of linear equations that posseses a decentralized character and can be used as a basis for routing decisions.

In [35] it is shown that there exist shadow prices associated with every link and route in the network using the fixed routing scheme explained. It is also shown how this shadow prices can be used for decision making for routing.

Following the notation defined in the previous sections for a network using fixed routing and the implied costs, it is shown in [35] that the derivatives of the network rate of return, W, with respect to external arrival rates, λ_r, and capacities, C_k, of the links are given by

$$\frac{dW(\underline{\lambda}, \mathrm{C})}{d\lambda_r} = (1 - L_r)\left(w_r - \sum_{k \in r} c_k\right), \tag{2.7}$$

$$\frac{dW(\underline{\lambda}, \mathrm{C})}{dC_k} = c_k. \tag{2.8}$$

Equation (2.7) shows the effect of increasing the exogenous traffic to route r, and it says that with probability $(1 - L_r)$ a call will be accepted on route r with a revenue of w_r and a cost for carrying the call of $\sum_{k \in r} c_k$. From equation (2.8) it can be seen that the costs c_k have an interpretation as shadow prices which measure the sensitivity of the network rate of return to the capacity C_k of link k.

Shadow prices have also been used in [12] where a state dependent routing strategy is presented which is based on revenue maximization, and where the control of revenue factors helps achieve network traffic maximization, priority for certain class of traffic and decrease in blocking for some type of traffic. In [14] shadow prices are used for multirate networks that use fixed routing. In Chapter 6 the shadow price methodology is extended to multirate networks with LLR.

2.2.4 Related Work

In [33], [16] and [31], the problem of the nonuniqueness of the solution for the fixed point equations of the distribution for the states of a link is treated thoroughly and an example is analyzed which we explain also in section 3.7.

In [16] the exchangeable link model is introduced which helps simplify the analysis of the multiple solutions of the fixed point equations. Also, the effect of trunk reservation is analyzed where there are multiple alternate routes available. The exchangeable link model considers a network where any OD pair can use any two links as an alternate route regardless of network topology.

In [33], the exchangeable link model for a network is generalized to consider graph structure. it is shown that respecting graph structure achieves the same results as those obtained with the exchangeable model. The problem of respecting graph structure is further treated in [11] where the same result is shown asymptotically. In [30] it is shown that least busy alternative routing with trunk reservation is asymptotically optimal. The routing policy considered is the least busy alternative with trunk reservation where an arriving call will be attempted on the direct single-link route, if there is no free capacity, then choose the least loaded two-link route from the alternate routes available, the call will be accepted if the trunk reservation parameter is satisfied otherwise it will be blocked and lost. In the selection of alternative routes ties are broken randomly. The model used is also the exchangeable link model. In this research, we show examples where MLLR performs better than LLR with no trunk reservation parameters when the network is highly asymmetric and where ties are broken by a predetermined sequence of routes.

In [34], a procedure for bounding dynamic routing schemes is described. The bound is developed from the theory of Markov decision processes as the solution of a convex programming problem where the mean flows are involved as well as the implied costs. In [17] the multiparented bound is introduced which is also mentioned in Section 3.6 of this dissertation.

2.3 Applications

In this section some applications are mentioned and in the following chapters developed using shadow prices. Shadow Prices can be used for network management, routing decisions [35], [36], network optimization [38], [70], [67] and pricing [52], [76]. Some of these applications are explained in Chapter 5 where some examples of the use of shadow prices such as sum capacity and pricing are shown. Also, shadow prices are used in a gradient descent algorithm to maximize the traffic carried in the network and keeping the OD pair blocking probability at certain prescribed level.

The framework given by the methodologies using shadow prices allows the fair comparison of different routing schemes through the concept of Sum Capacity, as well as the application of them to other kinds of telecommunication networks as it is seen in Chapter 7 with wireless networks determining the number of channels that have to be reserved for call hand-offs in each cell. Shadow prices can also be applied for pricing, where they reflect the cost of accepting a new call from certain OD pair.

In the following chapters, the analysis of LLR, ALBA and MLLR using shadow prices is presented, as well as the applications and extension of the results to multirate and wireless networks.

Chapter 3

Fixed Point Models

This chapter contains the description of the network model used in all the routing schemes analyzed, and the fixed point equations to evaluate performance such as network blocking or Origin-Destination blocking. Some results with figures containing such performance measures are introduced and compared among the schemes. The bounds used to quantify performance are also introduced, as well as the non-uniqueness of the solution to the fixed point equation of some routing schemes.

3.1 Model Description

In this section, the network, the arrival process and the call characteristics are explained. The notation introduced will be used in subsequent sections and chapters and can be consulted in Appendix B.

Consider a fully connected network. Let $\mathcal{N} = \{n_0, n_2, \ldots, n_{N-1}\}$ be the set of N nodes, $\mathcal{L} = \{(i,j) : 0 \leq i,j < N, i < j\}$, the ordered set of links (ordered in lexicographical order) and $\mathcal{O}$, the set of origin-destination (OD) pairs. Each OD pair $[i,j]$ ($0 \leq i,j < N$) has a set of routes, $\mathcal{R}_{ij}$, from which one is chosen to transmit an arriving call: these routes consist of the single-link direct route (the link (i,j) with capacity C_{ij}) and the set A_{ij} of two-link alternate routes. This choice of route set is typical of most adaptive schemes reported in the literature [54], [8], [58]. (In general, the set $\mathcal{R}_{ij}$ can be arbitrary and the results in the paper hold for arbitrary $\mathcal{R}_{ij}$; however, the numerical results presented in Section 4 assume route sets with a single link direct route and two-link alternate routes.) For an OD pair $[i, j]$, we denote by S_{ij} the set of links incident on the two nodes i and j. Figure 3.1 explicates some of this notation for a fully connected 4-node network.

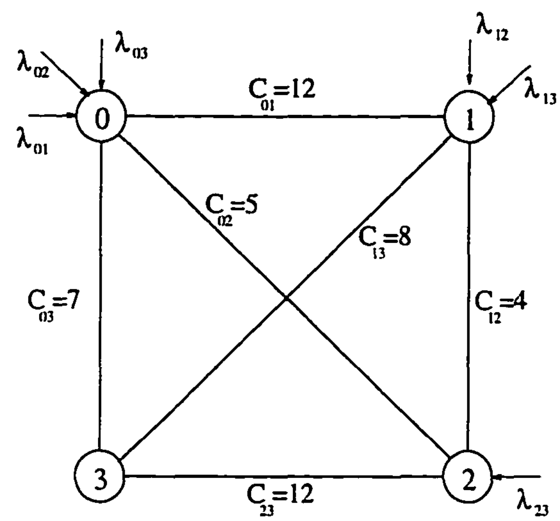

Figure 3.1: Fully Connected Four-Node Network

$$
\begin{aligned}
\mathcal{N} &= \{0, 1, 2, 3\}, \\
\mathcal{L} &= \{(0,1), (0,2), (0,3), (1,2), (1,3), (2,3)\}, \\
\mathcal{O} &= \{[0,1], [0,2], [0,3], [1,2], [1,3], [2,3]\}, \\
\mathcal{R}_{13} &= \{(1,3), (1,0,3), (1,2,3)\}.
\end{aligned}
$$

Links are assumed to be bidirectional and can transport traffic in either direction with equal capacity (thus $C_{ij} = C_{ji}$). For further reference, let $A_{ij}(ik) \in A_{ij}$ be the alternate route of OD pair $[i, j]$ that uses link (i, k) as one of the two links in the route. Call arrivals for OD pair $[i, j]$ form a Poisson process with rate λ_{ij} and these processes are assumed to be independent for different OD pairs. Consider a call arriving for OD pair $[i, j]$. If the call can be accommodated in a route chosen from the routing set $\mathcal{R}_{ij}$ according to the routing policy, then it is connected; otherwise it is blocked. Any call connected on one of the links requires the use of one unit of the available capacity, i.e., one trunk or one circuit. Call holding times are random variables with unit mean exponential distributions and are independent of earlier arrival times and holding times.

We assume that the occupancy of the trunks in link (i, j) evolves according to a birth-death process with the set of states $\{0, 1, 2, \ldots, C_{ij}\}$, independently of other links [8], [18], [54]. Let $p_{ij}(n)$ denote the probability that link (i, j) is in state n and let B_{ij} denote the blocking probability for OD pair $[i, j]$ (on account of link bidirectionality $p_{ij}(n) = p_{ji}(n)$ and $B_{ij} = B_{ji}$). As shown in [8], [54], the probabilities $p_{ij}(n)$ and B_{ij} can be obtained by using the fixed point method summarized in Sections 3.2 and 3.3. Finally, we let $\mathcal{T}_{ij}$ denote the set of *intermediate* nodes on the alternate routes for OD pair $[i, j]$.

In the following sections, the routing policies and additional notation are introduced for each of the routing schemes analyzed.

3.2 Least Loaded Routing (LLR)

Least Loaded Routing (LLR) [25], [8] is an adaptive routing scheme which tries to route new calls over the least occupied portions of the network. To describe LLR, we define the following additional notation. Let $A_{ij}(ik) \in A_{ij}$ be the alternate route of OD pair $[i, j]$ that uses link (i, k) as one of the two links in the route, T, the trunk reservation parameter and M_r, the number of free circuits on route r. The state of a link is chosen to be the number of free circuits; if the state is zero, then the link is fully occupied. The number of free circuits on an alternate route z, formed by links (i,j) with m_{ij} free circuits and (j,k) with m_{jk} free circuits, is determined as $M_z = \min\{m_{ij}, m_{jk}\}$.

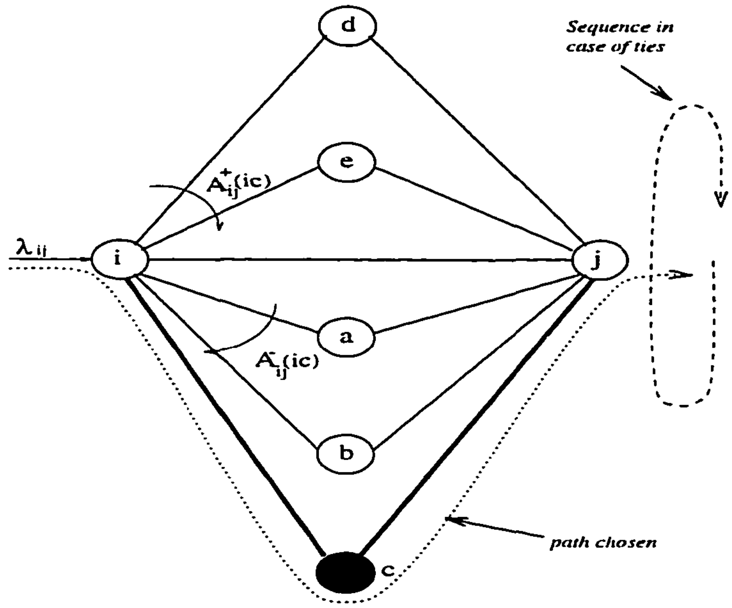

Figure 3.2: Model of Typical OD pair in a Network

$$\begin{aligned}
\mathcal{T}_{ij} &= \{a, b, c, d, e\}, \\
A_{ij} &= \{(i,a,j), (i,b,j), (i,c,j), (i,d,j), (i,e,j)\}, \\
A^-_{ij}(ic) &= \{(i,a,j), (i,b,j)\}, \\
A^+_{ij}(ic) &= \{(i,d,j), (i,e,j)\}.
\end{aligned}$$

The routing policy for LLR is as follows: when a call arrives for OD pair $[i,j]$, the call is first attempted on the direct link (i,j) – if there are free circuits available, then the call is accepted and routed on the direct route. If not, then the least loaded alternate route (i.e., the one with the largest number of free circuits) is considered. Suppose this alternate route is route z.

Then the call will be accepted and routed on route z if $M_z > T$; otherwise the call is blocked. For the case of two or more least loaded alternate routes, the ties are broken by a predetermined sequence in which the set of alternate routes is ordered: in this sequence $A^-_{jk}(ij) \subset A_{jk}$ is the set of alternate routes that *precede* $A_{jk}(ij)$ and $A^+_{jk}(ij) \subset A_{jk}$ is the set of alternate routes that *succeed* $A_{jk}(ij)$. Figure 3.2 explicates the notation for the sequence of alternate routes for OD pair $[i, j]$ with five alternate routes.

In [8], it is shown that a fixed point of the following system of equations provides a good approximation to the actual $p_{ij}(n)$'s .

$$p_{ij}(0) = \left\{1 + \sum_{m=1}^{C_{ij}} \left[\prod_{n=1}^{m} \frac{C_{ij} - n + 1}{\alpha_{ij}(n)}\right]\right\}^{-1}, \tag{3.1}$$

$$p_{ij}(m) = \left[\prod_{n=1}^{m} \frac{C_{ij} - n + 1}{\alpha_{ij}(n)}\right] p_{ij}(0), \quad m = 1, 2, \ldots, C_{ij}, \tag{3.2}$$

$$\alpha_{ij}(n) = \begin{cases} 0, & n = 0, \\ \lambda_{ij}, & 0 < n \leq T, \\ \lambda_{ij} + \sum_{b:[i,b]\in\mathcal{O}} \lambda_{ib} p_{ib}(0) \nu_{ib,j}(n) & \\ \quad + \sum_{b:[b,j]\in\mathcal{O}} \lambda_{bj} p_{bj}(0) \nu_{jb,i}(n), & T < n \leq C_{ij}. \end{cases} \tag{3.3}$$

where, $\alpha_{ij}(n)$ is the offered traffic to link (i, j) when it is in state n, (i.e., it has n free circuits); the arrival rate in state n is simply the number of calls connected on that link, i.e., $C_{ij} - n$.

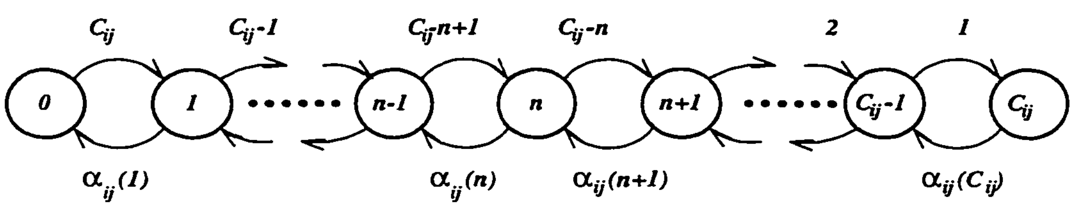

Figure 3.3: Birth-Death Process for link (i, j) using LLR

Figure 3.3 shows the birth-death process for link (i,j) using LLR where the birth rate is given by the number of calls present, and the death rate is given by the offered traffic calculated by equation (3.3). Both rates are state dependent where the state is given by the number of free circuits in the link. The terms $\nu_{ib,j}(n)$ and $\nu_{jb,i}(n)$ in equation (3.3) are the contributions to link (i,j) and (j,i) from OD pairs $[i,b]$ and $[j,b]$, respectively, when these OD pairs choose the two-link alternate route (i,j), (j,b) and (j,i), (i,b), respectively, when link (i,j) is in state n, and can be computed as follows

$$\begin{aligned}
\nu_{ib,j}(n) \;=\; & \sum_{m=T+1}^{n} p_{jb}(m) \prod_{d:(i,d),(d,b)\in A_{ib}^{-}(ij)} \left(1-\left[\sum_{s=m}^{C_{id}} p_{id}(s)\right]\left[\sum_{x=m}^{C_{db}} p_{db}(x)\right]\right) \\
& \cdot \prod_{d:(i,d),(d,b)\in A_{ib}^{+}(ij)} \left(1-\left[\sum_{s=m+1}^{C_{id}} p_{id}(s)\right]\left[\sum_{x=m+1}^{C_{db}} p_{db}(x)\right]\right) \\
& + \sum_{m=n+1}^{C_{jb}} p_{jb}(m) \prod_{d:(i,d),(d,b)\in A_{ib}^{-}(ij)} \left(1-\left[\sum_{s=n}^{C_{id}} p_{id}(s)\right]\left[\sum_{x=n}^{C_{db}} p_{db}(x)\right]\right) \\
& \prod_{d:(i,d),(d,b)\in A_{ib}^{+}(ij)} \left(1-\left[\sum_{s=n+1}^{C_{id}} p_{id}(s)\right]\left[\sum_{x=n+1}^{C_{db}} p_{db}(x)\right]\right), \qquad (3.4)
\end{aligned}$$

The term $\nu_{jb,i}(n)$ can be similarly defined and computed.

In Figure 3.4, it can be seen an example of the contribution to the offered traffic of a link due to overflow traffic from other OD pairs. The figure contains the sets of alternate routes that precede and succeed that two-link route of OD pair $[i,b]$ that uses link (i,j) in state n and link (j,b) in state m.

In equation (3.4), the first summation corresponds to the case $T < m \leq n$ which determines the state of the two-link route to be m. The second summation is for the case when the state of the two-link route is n, or $T < n \leq m$.

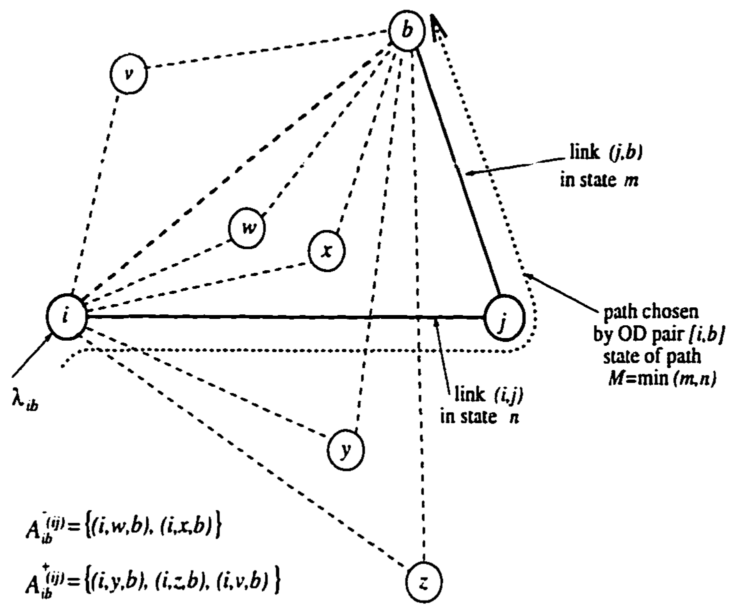

Figure 3.4: Contribution of Offered Traffic to Link (i,j) by OD pair $[i,b]$

Since the predefined sequence of routes for OD pair $[i,b]$ determines which route is chosen in case of ties, and the contribution of OD pair $[i,b]$ to link (i,j) is when the former chooses the two-link alternate route $A_{ib}(ij)$, we have that the product over $d : (i,d),(d,b) \in A^{-}_{ib}(ij)$ is over the routes of OD pair $[i,b]$ that precede the path (i,j), (j,b), and since this is the route that is chosen, the state of the preceding routes is taken such that their number of free circuits is less than that of route $A_{ib}(ij)$. Similarly for the product over $d : (i,d),(d,b) \in A^{+}_{ib}(ij)$, where the state of the preceding routes is considered when the number of free circuits is less than or equal to that of route $A_{ib}(ij)$. Finally, the OD pair blocking probability, B_{ij}, and the network blocking probability, L, are given by

$$B_{ij} = p_{ij}(0) \prod_{b \in T_{ij}} \left\{ 1 - \Big[\sum_{m=T+1}^{C_{ib}} p_{ib}(m) \Big] \Big[\sum_{m=T+1}^{C_{bj}} p_{bj}(m) \Big] \right\}, \tag{3.5}$$

$$L = \frac{\sum_{[i,j] \in \mathcal{O}} \lambda_{ij} B_{ij}}{\sum_{[i,j] \in \mathcal{O}} \lambda_{ij}}. \tag{3.6}$$

In Figure 3.5, it is shown the OD pair blocking probability for the four-node network of Figure 3.1 obtained by simulation and by the fixed point equations just presented. For this example, the external arrival rate for OD pair [1,2], i.e., λ_{12}, is varied, and the others are kept constant at a magnitude of equal to 75% of the direct link capacity as it can be seen in Table 3.1. It can be seen that the approximate blocking probability obtained by the fixed point equations is in closed agreement with that of the simulation. The figure also shows the influence of the trunk reservation parameter on the OD pair blocking, where for small traffic values, the probability becomes zero due to the fact that the links will not accept overflow traffic from other OD pairs when their state is on the reserved levels. For a discussion of the performance evaluation of this routing scheme, see [8].

Figure 3.7 has the comparison of simulation and numerical results for the five-node network of Figure 3.6. The remaining parameters are in Table 3.2 where it is shown that for an external arrival rate of 75% of the direct link capacity, the fixed point approximation is in close agreement with the simulation.

3.3 Aggregated Least Busy Alternative Routing (ALBA)

In [54], a routing algorithm, ALBA, which uses a reduced set of states, was introduced. In [54], the authors obtained the fixed point model for symmetric networks, in this thesis their idea is extended to asymmetric networks. In ALBA, the set of states for link (i,j) is divided into subsets each of which is then treated as an (aggregate) state for that link. Each link (i,j) has K aggregate states $\mathcal{A}_0^{ij}, \mathcal{A}_1^{ij}, \ldots, \mathcal{A}_{K-1}^{ij}$.

Table 3.1: Parameters for Simulation of Four-Node Network using LLR with $T = 2$, (λ_{12} was varied)

Simulation Parameters						
OD Pair	[0, 1]	[0, 2]	[0, 3]	[1, 2]	[1, 3]	[2, 3]
C_{ij}	12	5	7	4	8	12
λ_{ij}	9.0	3.75	5.25	*	6.0	9.0

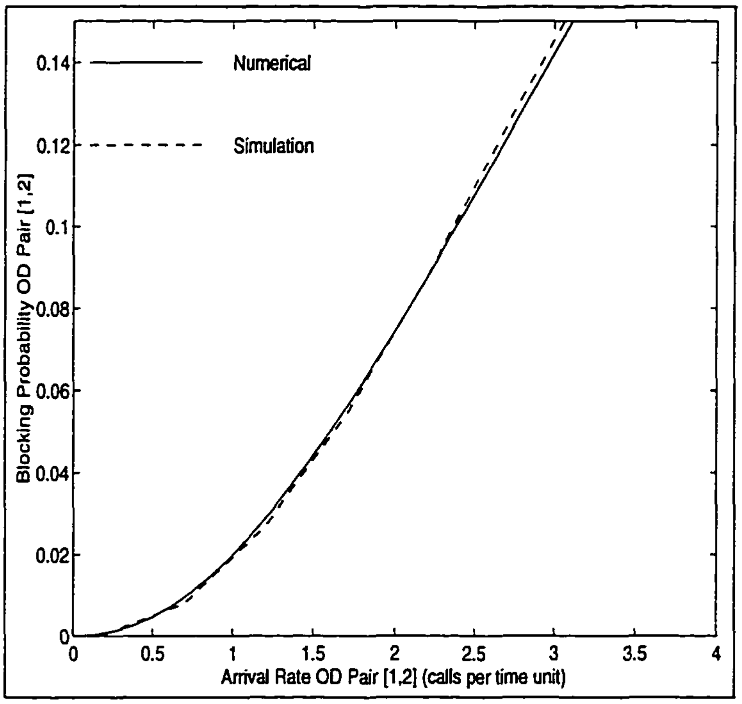

Figure 3.5: Simulated and Numerical OD pair Blocking probability for the Four-Node Network Using LLR with $T = 2$

Table 3.2: Parameters for Simulation of Five-Node Network using LLR with $T = 2$, (λ_{24} was varied)

Simulation Parameters										
OD Pair	[0, 1]	[0, 2]	[0, 3]	[0, 4]	[1, 2]	[1, 3]	[1, 4]	[2, 3]	[2, 4]	[3, 4]
C_{ij}	4	14	7	10	5	10	7	15	4	4
λ_{ij}	3.0	10.5	5.25	7.5	3.75	7.5	5.25	11.25	*	3

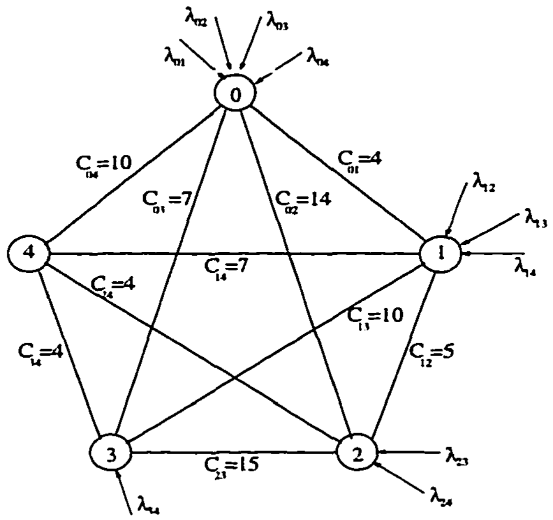

Figure 3.6: Five-node Network

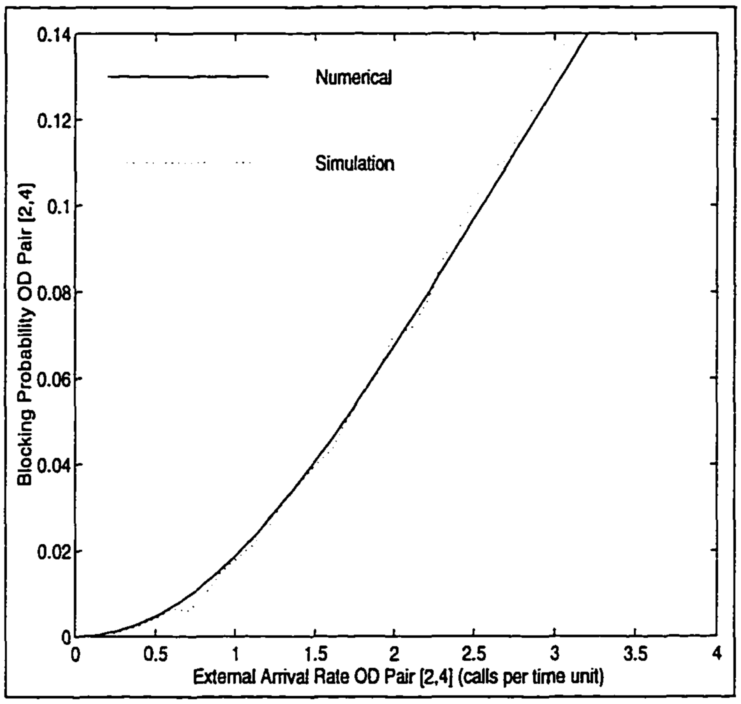

Figure 3.7: Simulated and Numerical OD pair Blocking probability for the Five-Node Network Using LLR with $T = 2$

For $m = 0, 1, 2, \ldots, K-2$, the m^{th} aggregate for link (i,j) is given by the set of states $\{C_{ij} - r_m^{ij}, C_{ij} - r_m^{ij} + 1, \ldots, C_{ij} - r_{m+1}^{ij} - 1\}$, where r_m^{ij} is some threshold value for link (i,j) and $r_0^{ij} = C_{ij} > r_1^{ij} > r_2^{ij} > \ldots, > r_{K-1}^{ij} > r_K^{ij} = 0$, the aggregate state $\mathcal{A}_{K-1}$ is given by the set of states $\{C_{ij} - r_{K-1}^{ij}, \ldots, C_{ij}\}$. If the two links of an alternate route, (i,j), (j,k) are in aggregate states $\mathcal{A}_l$ and $\mathcal{A}_m$, respectively, then the alternate route is said to be in aggregate state $\mathcal{A}_n$, where $n = \max\{l, m\}$.

The routing policy is similar to that of LLR. There exists preference for the single-link route. The arriving call is attempted on the direct link and it is accepted if there is capacity available; if not, then the alternate route with the smallest numbered aggregate state is found. The call is carried on this route if none of the links on this alternate route is in aggregate state $\mathcal{A}_{K-1}$. Otherwise, the call is rejected. Ties are again broken by a predetermined sequence in which the set of alternate routes is ordered: as in LLR, in this sequence $A_{jk}^-(ij) \subset A_{jk}$ is the set of alternate routes that *precede* $A_{jk}(ij)$ and $A_{jk}^+(ij) \subset A_{jk}$ is the set of alternate routes that *succeed* $A_{jk}(ij)$.

The fixed point algorithm is defined below, where we need some definitions for the probabilities of the aggregate states. Let $P_{ij}(I)$ denote the probability of link (i,j) being in aggregate state I, let $\Phi_{ij}^d(I)$ denote the probability that the alternate route (i,d), (d,j) of OD pair $[i,j]$ is in aggregate state I, and let $\Psi_{ij}^d(I)$ denote the probability that the alternate route (i,d), (d,j) of OD pair $[i,j]$ is in aggregate state I or higher. These probabilities can be computed as follows

$$P_{ij}(I) = \sum_{m \in \mathcal{A}_I^{ij}} p_{ij}(m), \tag{3.7}$$

$$\begin{aligned}
\Phi_{ij}^{d}(I) &= P_{id}(I)\sum_{n=0}^{I} P_{dj}(n) + P_{dj}(I)\sum_{n=0}^{I-1} P_{id}(n) \qquad (3.8)\\
&= \left[\sum_{n=0}^{I} P_{id}(n)\sum_{m=0}^{I} P_{dj}(m)\right] - \left[\sum_{n=0}^{I-1} P_{id}(n)\sum_{m=0}^{I-1} P_{dj}(m)\right]\\
&= \left[\sum_{n=0}^{C_{id}-r_{I+1}^{id}-1} p_{id}(n)\sum_{m=0}^{C_{dj}-r_{I+1}^{dj}-1} p_{dj}(m)\right] - \left[\sum_{n=0}^{C_{id}-r_{I}^{id}-1} p_{id}(n)\sum_{m=0}^{C_{dj}-r_{I}^{dj}-1} p_{dj}(m)\right],\\
\Psi_{ij}^{d}(I) &= \sum_{n=I}^{K-1}\Phi_{ij}^{d}(n) = 1 - \left[\sum_{n=0}^{I-1} P_{id}(n)\sum_{m=0}^{I-1} P_{dj}(m)\right], \quad 1 \le I < K,\\
\Psi_{ij}^{d}(0) &= 1. \qquad (3.9)
\end{aligned}$$

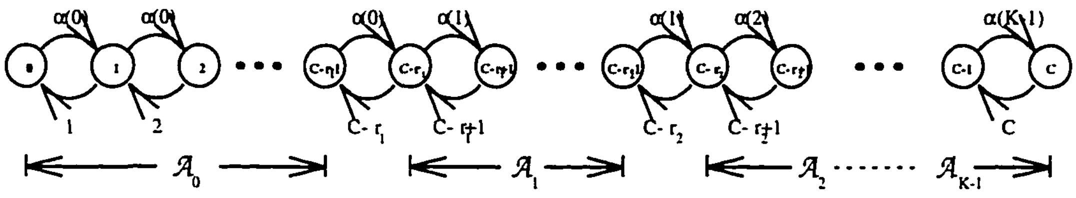

Figure 3.8: Birth-Death Process for a Link Using ALBA

The birth-death process for a link using ALBA with K aggregate states is in Figure 3.8. The birth and death rates are defined by the fixed point equations that follow. In [54], for the case of a symmetric network and below for asymmetric networks, it is shown that the fixed point of the following equations provides a good approximation to the actual $p_{ij}(n)$'s.

$$p_{ij}(0) = \left\{ \sum_{x=0}^{C_{ij}-r_1^{ij}} \frac{\alpha_{ij}^{x}(0)}{x!} + \sum_{u=1}^{K-1}\left[\prod_{n=0}^{u-1}\alpha_{ij}^{(r_n^{ij}-r_{n+1}^{ij})}(n)\right]\sum_{x=C_{ij}-r_u^{ij}+1}^{C_{ij}-r_{u+1}^{ij}} \frac{\alpha_{ij}^{x-(C_{ij}-r_u^{ij})}(u)}{x!}\right\}^{-1}, \qquad (3.10)$$

$$p_{ij}(m) = \frac{\alpha_{ij}^{m}(0)}{m!}p_{ij}(0), \qquad 0 < m \le C_{ij} - r_1^{ij}, \qquad (3.11)$$

$$p_{ij}(m) = \left[\prod_{n=0}^{u-1}\alpha_{ij}^{(r_n^{ij}-r_{n+1}^{ij})}(n)\right]\frac{\alpha_{ij}^{m-(C_{ij}-r_u^{ij})}(u)}{m!}p_{ij}(0), \qquad C_{ij} - r_u^{ij} < m \le C_{ij} - r_{u+1}^{ij},$$

$$u = 1, 2, ..., K-1, \qquad (3.12)$$

$$\alpha_{ij}(I) = \begin{cases} \lambda_{ij}, & I = K-1, \\ \lambda_{ij} + \sum\limits_{b:[i,b]\in\mathcal{O}} \lambda_{ib} p_{ib}(C_{ib}) \nu_{ib,j}(I) & \\ \quad + \sum\limits_{b:[b,j]\in\mathcal{O}} \lambda_{bj} p_{bj}(C_{bj}) \nu_{jb,i}(I), & 0 \le I < K-1. \end{cases} \tag{3.13}$$

In equation (3.13), the offered traffic to link (i,j) needs the terms $\nu_{ib,j}(I)$ and $\nu_{jb,i}(I)$ to be computed. These terms are the contributions to link (i,j) from those OD pairs which choose the two-link alternate route (i,j), (j,b) and (j,i), (i,b) respectively, when link (i,j) is in aggregate state I, and can be computed as follows

$$\begin{aligned} \nu_{ib,j}(I) &= \sum_{m=0}^{I} P_{bj}(m) \prod_{\substack{d:(i,d),(d,b)\in A_{ib}^{-}(ij) \\ d\neq j}} \Psi_{ib}^{d}(I+1) \prod_{\substack{d:(i,d),(d,b)\in A_{ib}^{+}(ij) \\ d\neq j}} \Psi_{ib}^{d}(I) \\ &+ \sum_{m=I+1}^{K-2} P_{bj}(m) \prod_{\substack{d:(i,d),(d,b)\in A_{ib}^{-}(ij) \\ d\neq j}} \Psi_{ib}^{d}(m+1) \prod_{\substack{d:(i,d),(d,b)\in A_{ib}^{+}(ij) \\ d\neq j}} \Psi_{ib}^{d}(m). \end{aligned} \tag{3.14}$$

The term $\nu_{jb,i}(I)$ can be similarly defined and computed.

Finally, the OD pair blocking probability, B_{ij}, and network blocking probability, L, are given by

$$B_{ij} = p_{ij}(C_{ij}) \prod_{b\in\mathcal{T}_{ij}} \left\{1 - [1 - P_{ib}(K-1)]\,[1 - P_{bj}(K-1)]\right\}, \tag{3.15}$$

$$L = \frac{\sum\limits_{[i,j]\in\mathcal{O}} \lambda_{ij} B_{ij}}{\sum\limits_{[i,j]\in\mathcal{O}} \lambda_{ij}}. \tag{3.16}$$

Next, some results of the comparison of ALBA and LLR are given where the performance measure is network blocking. The asymmetric networks investigated are diagramed in Figures 3.1 and 3.6 along with the corresponding capacities and exogenous traffic parameters. Symmetric networks, with 4 and 5 nodes with the capacity of each link being chosen to be the average capacity of the corresponding asymmetric network, were also investigated.

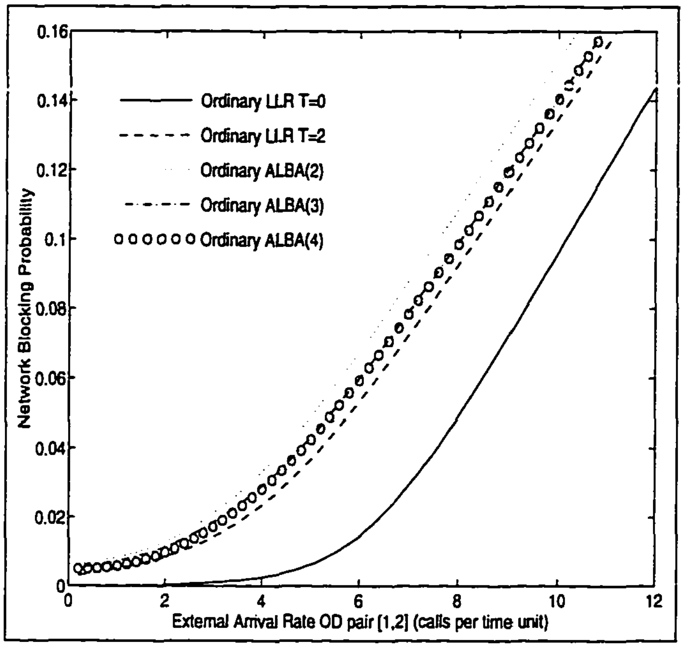

Figure 3.9: Network Blocking Probability for a Four-Node Network

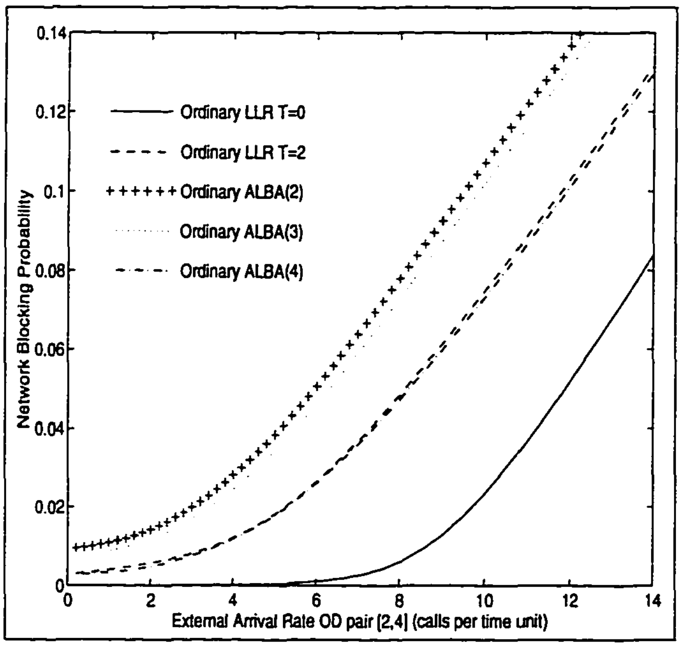

Figure 3.10: Network Blocking Probability for a Five-Node Network

Figures 3.9 and 3.10 show the network blocking against the same external arrival rate of the OD pair with least capacity in its direct single-link route. In all these figures the external arrival rate to the other OD pairs is kept fixed at a value of around .75 of the direct link capacity.

The network blocking probability very quickly rises to unacceptable values ($>$ .05) for arrival rates greater than 5 and 6 for the 4-node and 5-node networks, respectively. In all the cases, LLR without trunk reservation has the best performance. For larger values of K, ALBA(K) approaches the performance of LLR with trunk reservation and ALBA(4) has performance comparable to LLR with trunk reservation with $T = 2$. This is to be expected since ALBA has an inherent trunk reservation mechanism.

3.4 ALBA using Randomization (RALBA)

In this section, we present a modification of ALBA as presented above when making the decision of an alternate route in the case of ties. The notation for this routing scheme is the same as that used in Section 3.3 for ALBA. The difference between ALBA as presented in that section and that with randomization is in the routing policy when there is a tie in the number of alternate routes with smallest aggregate state, for RALBA, the alternate route will be chosen at random, whereas in Section 3.3 it was chosen by the predetermined sequence.

The birth-death process for RALBA is the same as that of Figure 3.8 where the birth rates are given by the offered traffic and are state dependent. The fixed point equations are given by the offered traffic to the links when these are at specific aggregate states and the state probability of the links. The offered traffic to link (i,j), when it is in aggregate state I, is defined in equation (3.13), but the terms for the contributions of all the OD pairs of the form $[i,b]$ and $[b,j]$, i.e., the terms $\nu_{ib,j}(I)$ and $\nu_{jb,i}(I)$, are given by

$$\begin{aligned}
\nu_{ib,j}(I) &= \sum_{H=0}^{K-2} P_{bj}(H) \\
&\cdot \left\{ \prod_{\substack{s\in\mathcal{T}_{ib}\\ s\neq j}} \Psi_{ib}^{s}(G+1) \right. \\
&+\frac{1}{2}\sum_{\substack{d\in\mathcal{T}_{ib}\\ d\neq j}} \Phi_{ib}^{d}(G) \prod_{\substack{s\in\mathcal{T}_{ib}\\ s\neq j,d}} \Psi_{ib}^{s}(G+1) \\
&+\frac{1}{3}\sum_{\substack{d\in\mathcal{T}_{ib}\\ d\neq j}} \Phi_{ib}^{d}(G) \sum_{\substack{e\in\mathcal{T}_{ib}\\ e\neq j,d}} \Phi_{ib}^{e}(G) \prod_{\substack{s\in\mathcal{T}_{ib}\\ s\neq j,d,e}} \Psi_{ib}^{s}(G+1) + \cdots \\
&\left. \cdots + \frac{1}{N-2} \prod_{\substack{s\in\mathcal{T}_{ib}\\ s\neq j}} \Phi_{ib}^{s}(G) \right\}. \qquad (3.17)
\end{aligned}$$

where $G = \max(I, H)$, $0 \leq G \leq K-1$, and the offered traffic to link (i,j) in aggregate state $K-1$ is $\alpha_{ij}(K-1) = \lambda_{ij}$. The term $\nu_{jb,i}(I)$ can be similarly defined and computed. The probabilities $p_{ij}(m)$ where m is the state of the link (i,j), i.e., the number of calls present in link (i,j), are given by equations (3.10), (3.11) and (3.12). With the blocking probability and the network blocking defined by equations (3.15) and (3.16), respectively.

The performance comparison of this routing scheme with LLR and ALBA in terms of the network blocking probability is shown in Figure 3.11 for the case of the 4-node network of previous examples. The figure compares the performance of LLR with and without trunk reservation ALBA(2), ALBA(3), ALBA(4), RALBA(2), RALBA(3) and RALBA(4).

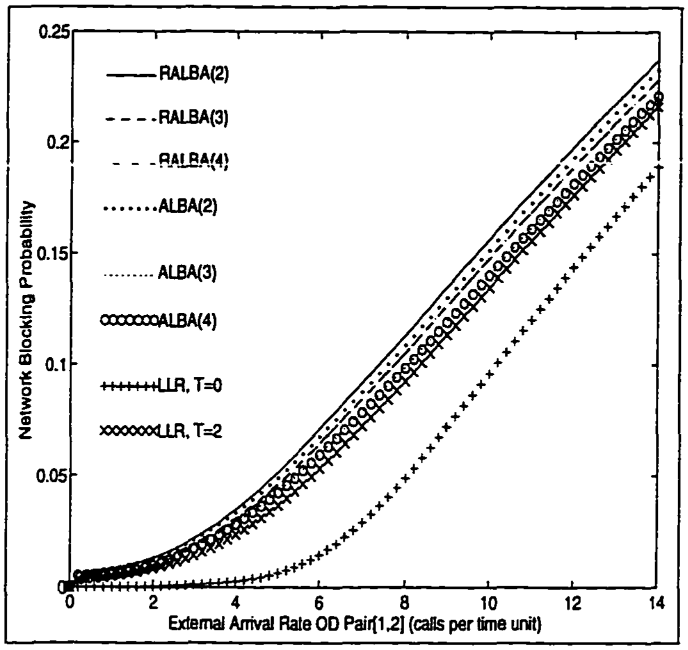

Figure 3.11: Network Blocking Probability for a Four-Node Network

The network blocking probability was obtained by the fixed point equations, keeping fixed all the external arrival rates at 75% of the direct link capacity except the one for OD pair $[1, 2]$. It can be seen that LLR with no trunk reservation ($T = 0$) performs better than the rest of the examples analyzed. RALBA(2) performs the worst and is followed by ALBA(2), RALBA(3) and RALBA(4) follow and have almost the same perfromance. From the figure we can deduct that ALBA, for the three examples, performs better than RALBA, and that LLR performs better than ALBA.

3.5 Modified LLR (MLLR)

MLLR is an adaptive routing scheme which tries to route new calls over the least occupied portions of the network. To describe this routing scheme, a single-link route formed by link (i, j) will be in state n when it has n free circuits, whereas the state of a two-link route formed by links (i, j) and (j, k), is determined as the minimum of the number of free circuits on each link. Let $\mathcal{Q}_{ij} = \{i,j\} \cap \mathcal{T}_{ij}$, i.e., the set of origin, destination and *intermediate* nodes for OD pair $[i,j]$.

The routing policy is as follows: Each OD pair $[i,j]$ has a number of available routes, and a distribution vector $\beta_{ij} = (\beta_{ij}(k) : k \in \mathcal{Q}_{ij}, 0 \leq \beta_{ij}(k) \leq 1, \sum_{k \in \mathcal{Q}_{ij}} \beta_{ij}(k) = 1)$, i.e., the load sharing coefficients, which determines the proportion of traffic allocated to each route. $\beta_{ij}(k)$ is the proportion of external arrival rate to OD pair $[i,j]$ allocated to route (i,k), (k,j). And $\beta_{ij}(j)$ or $\beta_{ij}(i)$ is the proportion allocated to the single-link route (i,j). The *direct* or *first-choice* route is chosen randomly on a per call basis with distribution β_{ij}, the other available routes at that moment are called *alternate* routes. Depending on the *first-choice* route selected by OD pair $[i,j]$, the available routes are ordered in a sequence beginning with the *first-choice* route. When a call arrives for OD pair $[i,j]$, the *first-choice* route is determined and the call is attempted on it, if this route has at least one free circuit, then the call is transmitted through it. If the *first-choice* route has no free circuits then the call will be routed through the least loaded path (i.e., one with the largest number of free circuits) chosen from the rest of the available routes. Ties are broken using the predetermined sequence of routes. Least Loaded Routing (LLR) described in [8] is *contained* by this routing scheme in the case when the value of β_{ij} is one for the single link route and zero for the two-link alternate routes, i.e., LLR has direct link preference. And using the same notation as that for LLR, the solution for the distribution of the states for the links, $p_{ij}(n)$, $n = 0, 1, \ldots, C_{ij}$ can be obtained from the following fixed point model

$$p_{ij}(0) = \left\{1 + \sum_{m=1}^{C_{ij}} \left[\prod_{n=1}^{m} \frac{C_{ij} - n + 1}{\alpha_{ij}(n)}\right]\right\}^{-1}, \tag{3.18}$$

$$p_{ij}(m) = \left[\prod_{n=1}^{m} \frac{C_{ij} - n + 1}{\alpha_{ij}(n)}\right] p_{ij}(0), \quad m = 1, 2, \ldots, C_{ij}, \tag{3.19}$$

$$\alpha_{ij}(m) = \begin{cases} 0, & m = 0, \\ \lambda_{ij}\beta_{ij}(j) + \sum\limits_{\substack{b:[i,b]\in\mathcal{O} \\ b\neq j}} \lambda_{ib}\beta_{ib}(j)\left[1 - p_{jb}(0)\right] & \\ \quad + \sum\limits_{\substack{b:[b,j]\in\mathcal{O} \\ b\neq i}} \lambda_{bj}\beta_{bj}(i)\left[1 - p_{ib}(0)\right] & \\ \quad + \sum\limits_{b:[i,b]\in\mathcal{O}} \sum\limits_{\substack{d\in S_{ib} \\ d\neq j}} \lambda_{ib}\beta_{ib}(d) P_{ib}(d,j,m) & \\ \quad + \sum\limits_{\substack{b:[b,j]\in\mathcal{O} \\ b\neq i}} \sum\limits_{\substack{d\in S_{bj} \\ d\neq i}} \lambda_{bj}\beta_{bj}(d) P_{bj}(d,i,m), & m > 0, \end{cases} \tag{3.20}$$

where for $b \in \mathcal{T}_{ij}$, and $d \in \mathcal{T}_{ib}$ we have $P_{ib}(d,j,m)$ as the probability that OD pair $[i,b]$ chooses alternate route (i,j), (j,b) with link (i,j) in state m, when *first-choice* route of OD pair $[i,b]$ is the path (i,d), (d,b). If $d = b$ then the *first-choice* route for OD pair $[i,b]$ is the single-link route (i,b), giving $P_{ib}(b,j,m)$. Similarly for $P_{bj}(d,i,m)$ and $P_{bj}(j,i,m)$. Hence, for $d \neq b$ we get

$$\begin{aligned} P_{ib}(d,j,m) = \; & \left\{ 1 - \left[1 - p_{id}(0)\right]\left[1 - p_{db}(0)\right]\right\} \\ & \left\{ \sum_{n=1}^{m} \left[p_{bj}(n)\left(1 - \sum_{\substack{w=n \\ b\in A^{-}_{ib}(ij)}}^{C_{ib}} p_{ib}(w)\right)\left(1 - \sum_{\substack{w=n+1 \\ b\in A^{+}_{ib}(ij)}}^{C_{ib}} p_{ib}(w)\right) \right.\right. \\ & \cdot \prod_{\substack{u:(i,u),(u,b)\in A^{-}_{ib}(ij) \\ u\neq b,d}} \left(1 - \left[\sum_{s=n}^{C_{iu}} p_{iu}(s)\right]\left[\sum_{x=n}^{C_{ub}} p_{ub}(x)\right]\right) \\ & \left. \cdot \prod_{\substack{u:(i,u),(u,b)\in A^{+}_{ib}(ij) \\ u\neq b,d}} \left(1 - \left[\sum_{s=n+1}^{C_{iu}} p_{iu}(s)\right]\left[\sum_{x=n+1}^{C_{ub}} p_{ub}(x)\right]\right)\right] \\ & + \sum_{n=m+1}^{C_{bj}} p_{bj}(n)\left(1 - \sum_{\substack{w=m \\ b\in A^{-}_{ib}(ij)}}^{C_{ib}} p_{ib}(w)\right)\left(1 - \sum_{\substack{w=m+1 \\ b\in A^{+}_{ib}(ij)}}^{C_{ib}} p_{ib}(w)\right) \end{aligned}$$

$$\cdot \prod_{\substack{u:(i,u),(u,b)\in A^-_{ib}(ij)\\ u\neq b,d}} \left(1-\left[\sum_{s=m}^{C_{iu}} p_{iu}(s)\right]\left[\sum_{x=m}^{C_{ub}} p_{ub}(x)\right]\right)$$

$$\cdot \prod_{\substack{u:(i,u),(u,b)\in A^+_{ib}(ij)\\ u\neq b,d}} \left(1-\left[\sum_{s=m+1}^{C_{iu}} p_{iu}(s)\right]\left[\sum_{x=m+1}^{C_{ub}} p_{ub}(x)\right]\right)\Bigg\}, \tag{3.21}$$

and for $d = b$ we have

$$\begin{aligned} P_{ib}(b,j,m) = \quad & p_{ib}(0)\Bigg\{\sum_{n=1}^{m}\Bigg[p_{bj}(n) \prod_{\substack{u:(i,u),(u,b)\in A^-_{ib}(ij)\\ u\neq b}} \left(1-\left[\sum_{s=n}^{C_{iu}} p_{iu}(s)\right]\left[\sum_{x=n}^{C_{ub}} p_{ub}(x)\right]\right) \\ & \cdot \prod_{\substack{u:(i,u),(u,b)\in A^+_{ib}(ij)\\ u\neq b}} \left(1-\left[\sum_{s=n+1}^{C_{iu}} p_{iu}(s)\right]\left[\sum_{x=n+1}^{C_{ub}} p_{ub}(x)\right]\right)\Bigg] \\ & + \sum_{n=m+1}^{C_{bj}} p_{bj}(n) \prod_{\substack{u:(i,u),(u,b)\in A^-_{ib}(ij)\\ u\neq b}} \left(1-\left[\sum_{s=m}^{C_{iu}} p_{iu}(s)\right]\left[\sum_{x=m}^{C_{ub}} p_{ub}(x)\right]\right) \\ & \cdot \prod_{\substack{u:(i,u),(u,b)\in A^+_{ib}(ij)\\ u\neq b}} \left(1-\left[\sum_{s=m+1}^{C_{iu}} p_{iu}(s)\right]\left[\sum_{x=m+1}^{C_{ub}} p_{ub}(x)\right]\right)\Bigg\}. \end{aligned} \tag{3.22}$$

For the case when $b = j$, and $b \neq d$, i.e., $P_{ij}(d,j,m)$, we get the probability that OD pair $[i,j]$ chooses the single-link route (i,j) when its *first-choice* route is the path (i,d), (d,j) which is given by

$$\begin{aligned} P_{ij}(d,j,m) = \; & \Big\{\; 1-\big[1-p_{id}(0)\big]\big[1-p_{dj}(0)\big]\Big\} \\ & \cdot \prod_{\substack{u:(i,u),(u,j)\in A^-_{ij}(ij)\\ u\neq j,d}} \left(1-\left[\sum_{s=m}^{C_{iu}} p_{iu}(s)\right]\left[\sum_{x=m}^{C_{ub}} p_{uj}(x)\right]\right) \\ & \cdot \prod_{\substack{u:(i,u),(u,j)\in A^+_{ij}(ij)\\ u\neq j,d}} \left(1-\left[\sum_{s=m+1}^{C_{iu}} p_{iu}(s)\right]\left[\sum_{x=m+1}^{C_{ub}} p_{uj}(x)\right]\right). \end{aligned} \tag{3.23}$$

In Figure 3.12, it can be seen an example of the contribution to the offered traffic of a link due to overflow traffic from other OD pairs. The blocking probability for OD pair $[i,j]$, B_{ij}, and the network blocking probability, L, are given by

$$B_{ij} = p_{ij}(0) \prod_{b \in T_{ij}} \left\{ 1 - \left[1 - p_{ib}(0)\right]\left[1 - p_{bj}(0)\right] \right\}, \tag{3.24}$$

$$L = \frac{\sum_{[i,j] \in \mathcal{O}} \lambda_{ij} B_{ij}}{\sum_{[i,j] \in \mathcal{O}} \lambda_{ij}}. \tag{3.25}$$

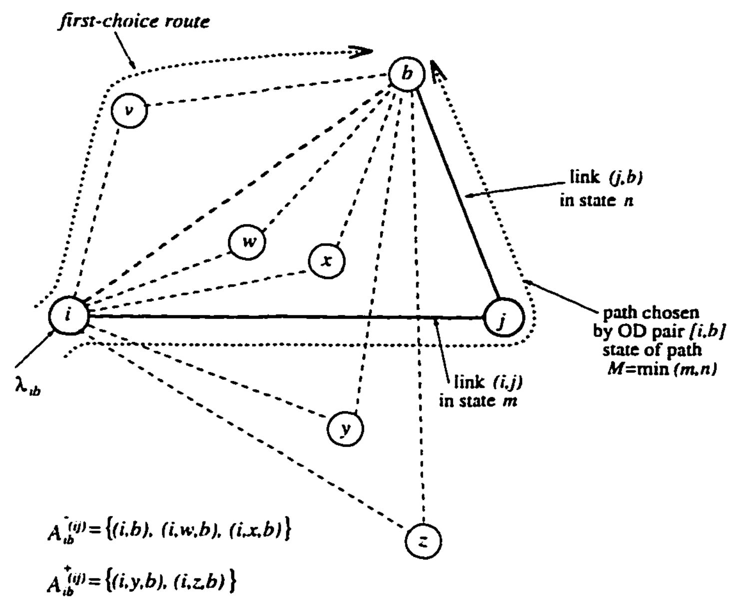

Figure 3.12: Contribution to Offered Traffic to Link (i,j) from OD Pair $[i,b]$ when Path (i,v), (v,b) is the *First-Choice* Route

Results concerning the network blocking probability for MLLR are similar to those of LLR. MLLR is used in Chapter 5 maximizing traffic and comparing OD pair blocking and carried traffic with those of LLR.

Among all the routing schemes in existence, there is no point of comparison with an *optimal* routing scheme, hence performance comparison has to be based on bounds on the blocking probability and the carried traffic in the network. These bounds are introduced in the next section, and in chapters 4 and 5 used for the carried traffic and optimization, respectively.

3.6 Bounds on the Performance

In this section the bounds used to evaluate the performance of the routing schemes just explained are introduced. These bounds are the Erlang bound and Max-Flow bound found in [18] and the Single Parented bound found in [17]. The Max-Flow bound is extended to dynamic routing in [34].

3.6.1 Erlang Bound

[18] introduces the Erlang bound as a solution of a minimization problem that we explain in the following. Given Erlang's formula $E(\lambda, C)$, this bound considers the entire network as a link with capacity $C = \sum_{(i,j)\in\mathcal{L}} C_{ij}$ and offered traffic $\lambda = \sum_{[i,j]\in\mathcal{O}} \lambda_{ij}$, and represents the minimum network blocking probability, L, for such link. The bound is obtained by finding the solution to the following minimization problem.

$$\begin{aligned} \min_{B_{ij}} \quad & L = \frac{\sum_{[i,j]\in\mathcal{O}} \lambda_{ij} B_{ij}}{\sum_{[i,j]\in\mathcal{O}} \lambda_{ij}} \\ \text{subject to} \quad & \sum_{[i,j]\in\mathcal{O}} \lambda_{ij} B_{ij} \geq \left(\sum_{[i,j]\in\mathcal{O}} \lambda_{ij} \right) E\Big(\sum_{[i,j]\in\mathcal{O}} \lambda_{ij},\ \sum_{(i,j)\in\mathcal{L}} C_{ij} \Big) \\ & B_{ij} \geq 0, \quad \forall\, [i,j] \in \mathcal{O}. \end{aligned} \tag{3.26}$$

where B_{ij} is the blocking probability of link (i, j). The solution to this minimization problem gives as result the blocking probabilities for each link such that for given external arrival rates the network blocking is minimum.

3.6.2 The Max-Flow Bound

This bound is introduced in [18] and represents the total flow through the network with capacity and offered traffic constraints. The constraints are determined by the total offered traffic to each link and the capacity constraints where the total flow on each link cannot exceed its capacity. Let x_{ij} be the flow on link (i,j) due to single-link route traffic and x_{ikj} the flow through the two-link path (i,k), (k,j), then the Max-Flow bound is given by the solution of the following constrained linear programming maximization problem.

$$\begin{aligned}
\max_{\underline{x}} \quad & \sum_{[i,j]\in\mathcal{O}} \left(x_{ij} + \sum_{k\in\mathcal{T}_{ij}} x_{ikj} \right) \\
\text{subject to} \quad & x_{ij} + \sum_{k\in\mathcal{T}_{ij}} x_{ikj} \leq \lambda_{ij}, \quad \forall\, [i,j] \in \mathcal{O} \\
& x_{ij} + \sum_{k\in\mathcal{T}_{ij}} (x_{ijk} + x_{jik}) \leq C_{ij}, \quad \forall\, (i,j) \in \mathcal{L} \\
& x_{ij} \geq 0, \ x_{ijk} \geq 0, \ x_{ikj} = x_{jki}, \ \forall\, i,j,k \in \mathcal{N}.
\end{aligned} \tag{3.27}$$

where $\underline{x}$ is the vector of x_{ij}'s and x_{ijk}'s $\forall\, i,j,k \in \mathcal{N}$.

3.6.3 Single Parented Bound

Following the introduction of this bound on [17] we have that it is obtained from a model which is described as follows. Consider a fully connected network where each link (i,j) with capacity C_{ij} is offered Poisson traffic at rate λ_{ij} and a second stream at infinite rate. The holding times for those accepted calls are independent and identically distributed with an exponential distribution and unit mean. The rewards generated by calls accepted from the Poisson traffic source are 1, while for the infinite rate source are $\frac{1}{2}$. Let $\pi_{ij}(n)$ be the stationary probability that link (i,j) is in state n, where n is the number of calls present in that link, then the long run average reward earned per unit time cannot exceed

$$\sum_{[i,j]\in\mathcal{O}} \left[\lambda_{ij}(1 - \pi_{ij}(C_{ij})) + \frac{1}{2}(C_{ij} - r_{ij})\pi_{ij}(C_{ij} - r_{ij}) \right], \tag{3.28}$$

where r_{ij} is the trunk reservation parameter to give priority to the Poisson traffic. The bound is obtained as the solution of an optimization problem by choosing the set of parameters r_{ij}, $\forall\ (i,j) \in \mathcal{L}$ such that (3.28) is maximized or such that the network blocking which is given by

$$L = \frac{\left\{ \sum_{[i,j]\in\mathcal{O}} \left[\lambda_{ij}\pi_{ij}(C_{ij}) - \frac{1}{2}(C_{ij} - r_{ij})\pi_{ij}(C_{ij} - r_{ij}) \right] \right\}^{+}}{\sum_{[i,j]\in\mathcal{O}} \lambda_{ij}}, \tag{3.29}$$

is minimized. This bound is applied to any routing policy regardless of trunk reservation or two-link paths.

3.7 Non-Uniqueness of Blocking Probabilities

In this section, we show the non-uniqueness of the LLR model by comparing it to the example of Kelly in [31] for Fixed Alternate Routing (FAR). A discussion on the non-uniqueness and an example is presented in [2]. The example by Kelly is as follows. Suppose that we have a symmetric fully connected network with at least 3 nodes, each link has capacity C and every OD pair has Poisson arrivals with rate λ. The routing policy is

- When a call arrives, accept it in the direct single-link route if there is free capacity.
- When there is no capacity available in the direct link, choose at random a two-link alternate route, if there is capacity available on both links then accept the call, otherwise the call will be rejected.

This routing is dynamic with one alternate route for every OD pair. Let B be the link blocking probability (equal for all links).

Following Erlang fixed point approximation to calculate the blocking probability for every link, we need to obtain the offered traffic first. We have that the offered traffic to every link is $\lambda\left[1+2B\left(1-B\right)\right]$, hence we have the fixed point equation

$$B = E\left(\lambda\left[1+2B\left(1-B\right)\right],\ C\right), \tag{3.30}$$

where $E(\cdot,\cdot)$ is the Erlang B formula. Using the limiting regime in [31], replace λ by λN and C by CN and letting N go to ∞, we have the following approximation

$$E\left(\lambda N,\ CN\right) \to \max\left\{0,\ 1-\frac{C}{\lambda}\right\}, \tag{3.31}$$

uniformly on $\lambda \in [0, \infty)$, hence equation (3.30) becomes

$$B = \max\left\{0,\ 1-\frac{C}{\lambda\left[1+2B\left(1-B\right)\right]}\right\}. \tag{3.32}$$

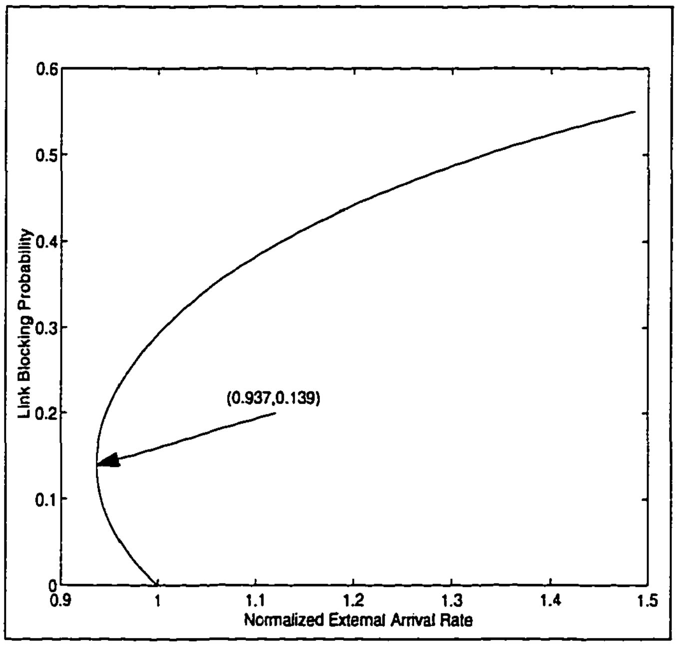

Figure 3.13: Locus of Points for Link Blocking Probability using FAR

The locus of points satisfying this last equation is shown in Figure 3.13. The turning point on the left of the curve is $(\lambda^{*},\ B^{*}) = (0.937C,\ 0.140)$, and for $\lambda \in$

(λ^*, C) there exist three solutions to equation (3.32). To obtain the values of the coordinates of the turning point we do the following. Let $y = \lambda/C$. Define $F(y, B)$ as follows.

$$F(y,B) = B - 1 + \frac{1}{y\,[1 + 2B\,(1-B)]}. \tag{3.33}$$

Apply the Implicit Function Theorem, [15], to $F(y, B)$ as follows. Obtain $\partial F(y,B)/\partial B = 0$ to get

$$\frac{\partial F(y,B)}{\partial B} = 1 + \frac{2B - 2(1-B)}{[1 + 2B\,(1-B)]^2} = 0, \tag{3.34}$$

solve for y to obtain

$$y = \frac{2 - 4B}{[1 + 2B\,(1-B)]^2}. \tag{3.35}$$

From (3.33) get $F(y, B) = 0$ to obtain

$$B = 1 - \frac{1}{y\,[1 + 2B\,(1-B)]}, \tag{3.36}$$

solve for y to obtain

$$y = \frac{1}{[1 + 2B\,(1-B)]\,(1-B)}. \tag{3.37}$$

Equate (3.35) and (3.37) and solve for B to get $B = 0.139619$, substitute this value into (3.37) to find $y = 0.937$ or, by using the identity $y = \lambda/C$, $\lambda = 0.937C$ as stated before. Now, we proceed to apply the same procedure to LLR as follows.

Consider a symmetric and fully connected 3-node network using LLR. From the fixed point equations (3.1), (3.2) and (3.3), and the fact that for symmetric networks all the links have the same stationary distribution for the states, the notation for the model can be simplified by considering the following identities $p_{ij}(n) = p_n$, $\alpha_{ij}(n) = \alpha$, $\forall n$, $C_{ij} = C$, $\lambda_{ij} = \lambda\ \forall (i,j)$.

The corresponding expressions for the probability of a link being fully occupied using these identities are

$$p_0 = \left\{1 + \sum_{i=1}^{C} \frac{C!}{\alpha^i (C-i)!}\right\}^{-1}, \tag{3.38}$$

where only p_0 is needed to obtain the locus of points desired, since p_0 is the link blocking probability, then the offered traffic from (3.3) and (3.4) is given by

$$\alpha = \lambda \left[1 + 2p_0(1-p_0)\right], \tag{3.39}$$

which is the same as the offered traffic to every link in the example just explained above. From (3.38) it can be obtained the following expression

$$\frac{\alpha^C}{C!} - p_0 \sum_{k=0}^{C} \frac{\alpha^k}{k!} = 0, \tag{3.40}$$

and from (3.39) we can obtain

$$\frac{\lambda}{C} = \frac{\alpha/C}{1 + 2p_0(1-p_0)}. \tag{3.41}$$

The objective is to obtain the locus of points for p_0 as a function of the normalized arrival rate $\frac{\lambda}{C}$. The following algorithm gets those points

- Consider that p_0 can take any value in $[0, 1]$.
- Given p_0 and C, find the value of $\alpha \in (0, \infty)$ that satisfies (3.40).
- With p_0, C and α obtain the value of λ/C using (3.41).
- Repeat for other values of p_0.

Now, the turning points for this example can be found by applying the Implicit Function Theorem as in the previous example and that we describe as follows.

We have to define the functions to be used in the Implicit Function Theorem in the substitution and derivation. They are

$$p_0(\alpha) = \frac{\alpha^C/C!}{\sum_{k=0}^{C} \frac{\alpha^k}{k!}}, \tag{3.42}$$

$$f(\lambda, \alpha) = \frac{\alpha}{1 + 2p_0(\alpha)(1 - p_0(\alpha))} - \lambda. \tag{3.43}$$

Next, apply the Implicit Function Theorem to (3.43) to find the point or points on which the Jacobian vanishes, i.e., consider $f(\lambda, \alpha) = 0$, which gives (3.41), and also consider $\partial f(\lambda, \alpha)/\partial \alpha = 0$ which is the condition when the Jacobian vanishes. Find the value of α that satisfies the partial derivative, and with it, find the value of p_0 using (3.42) and take those two values to (3.41) to obtain the λ, then the point or points where the Jacobian vanishes will be given by the triplets (λ, α, p_0) obtained.

The fixed point models presented in this chapter for LLR, ALBA, RALBA and MLLR are used in the next chapter to obtain the shadow prices by defining the network rate of return using the OD pair blocking probability B_{ij}'s. The derivatives with respect to the external arrival rates of these probabilities are calculated since they are implicit functions of the exogenous arrivals.

Chapter 4

Shadow Prices

In this chapter, we present the shadow price methodology and some applications for the routing schemes explained in the previous chapter. In [35], the author calculates shadow prices, i.e., the derivative of the rate of return of the network with respect to exogenous traffic and with respect to link capacities, for FAR. In [36], this work is extended to include the case of FAR with trunk reservation and the author suggests how the method could be extended to adaptive routing schemes. Shadow prices can be used in a number of applications such as in algorithms to aid capacity expansion decisions [22], [59], in pricing policy [52], for the apportionment of revenue between various sections of a network, [76], and to derive quasi-static adaptive routing schemes from FAR [35], [36].

We calculate shadow prices with respect to exogenous traffic for LLR, ALBA and RALBA, and with respect to the load sharing coefficients for MLLR. In these calculations, the fixed point algorithms for performance evaluation of LLR in Section 3.2, ALBA in Section 3.3 and RALBA in Section 3.4, are used as a starting point. Shadow prices are then utilized in the comparative analysis of adaptive routing schemes, in matching demand and capacity distribution, and in traffic maximization in such schemes.

4.1 Shadow Price Formulation

Let $\underline{\lambda}$ denote the vector of exogenous traffic offered to OD pairs, and let w_{ij} denote the revenue generated by accepting a call on OD pair $[i, j]$. (When blocking probability is the performance measure considered, all the w_{ij}'s can be chosen to be 1.) Then, the network rate of return, W, can be written as

$$W(\underline{\lambda}, \underline{\mathcal{B}}) = \sum_{[j,k]\in\mathcal{O}} \lambda_{jk} w_{jk} \left(1 - B_{jk}(\underline{p})\right), \tag{4.1}$$

where $\underline{p}$ is a vector obtained by the concatenation of the equilibrium probability vectors of each link and $\underline{\mathcal{B}}$ is the vector of OD pair blocking probabilities. (In (4.1), the network rate of return is written as $W(\underline{\lambda}, \underline{\mathcal{B}})$ to show its explicit dependence on $\underline{\lambda}$ and $\underline{\mathcal{B}}$. Note that since the equilibrium probability vectors and blocking probabilities we use are those given by the fixed point models of Chapter 3, the W in (4.1) is an *approximate* rate of return. As a result, the shadow prices we calculate are also approximations to the actual shadow prices. However, we also calculate shadow prices from simulations which verify the accuracy of the approximation.

We now use (4.1) and the fixed-point models of Chapter 3 to calculate the shadow prices of LLR, ALBA(K) and RALBA, namely the derivatives of the rate of return with respect to exogenous traffic.

The fixed point models describe the $\underline{p}$, as an implicit function of $\underline{\lambda}$. $\underline{\mathcal{B}}$ is, in turn, a function of $\underline{p}$ and thereby an implicit function of $\underline{\lambda}$. Consequently, $W(\underline{\lambda}, \underline{\mathcal{B}})$ is also an implicit function of $\underline{\lambda}$. We therefore undertake a careful and extensive effort to obtain relations of total and partial derivatives of OD pair blocking probabilities by differentiating the fixed point equations. These relations are manipulated to obtain a system of linear equations in the derivatives of the OD pair blocking probabilities with respect to exogenous traffic.

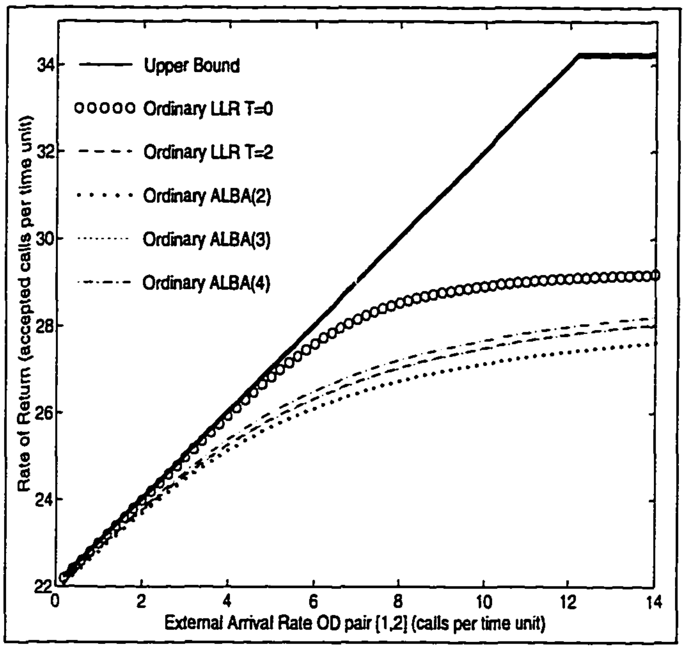

Figure 4.1: Rate of Return and Bounds for a Four-Node Network

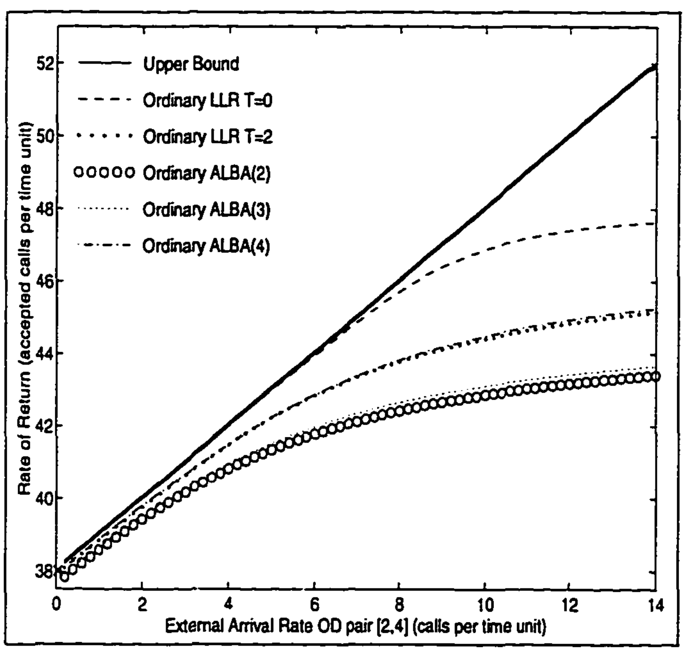

Figure 4.2: Rate of Return and Bounds for a Five-Node Network

Figures 4.1 and 4.2 show the rate of return using LLR, LLR with trunk reservation, ALBA(2), ALBA(3) and ALBA(4) for the 4-node network and the 5-node network, respectively, against the external arrival rate for the OD pair with the smallest direct link capacity (i.e., OD pairs $[1,2]$ and $[2,4]$ for the 4-node and 5-node networks, respectively). Figures 3.9 and 3.10 show the network blocking versus the same external arrival rate. In all these figures the external arrival rate to the other OD pairs is kept fixed at a value of around 75% of the direct link capacity. The figures reveal that, although the rate of return rises quite rapidly with the external arrival rate, the corresponding network blocking probability very quickly rises to unacceptable values ($> .05$) for arrival rates greater than 5 and 6 for the 4-node and 5-node networks, respectively.

In all the cases, LLR without trunk reservation has the best performance. For larger values of K, ALBA(K) approaches the performance of LLR with trunk reservation and ALBA(4) has performance comparable to LLR with trunk reservation with $T = 2$. This is to be expected since ALBA has an inherent trunk reservation mechanism. Figures 4.1 and 4.2 also show upper bounds on the rate of return obtained from the max-flow and Erlang bounds of [18] and the single-parented bound of [17] which are explained in Section 3.6.

The figures indicate that for all the networks we consider, for the blocking probabilities of interest, LLR is nearly optimal in terms of the rate of return for this configuration of external arrival rates. The network rate of return for the same 4-node network, but using RALBA as well, is shown in Figure 4.3. It can be seen that

ALBA performs better than RALBA, meaning that a sequence of routes is lightly better than randomized routing.

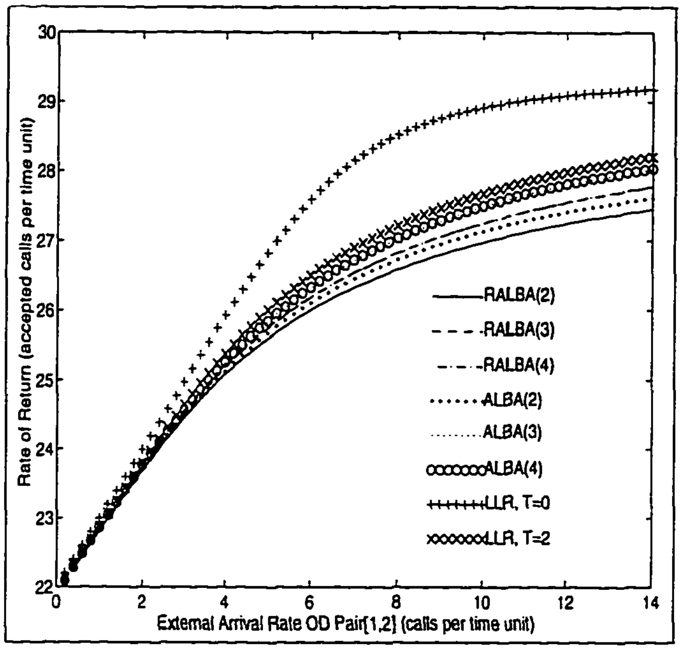

Figure 4.3: Rate of Return for a Four-Node Network

It is known that the blocking probability solutions to the fixed point equations for alternate routing may not be unique [20], [35], see also Section 3.7. Even in this case, under a maximum rank condition on the Jacobian of the fixed point equations, the implicit function theorem [15], confirms the existence of a locally differentiable function whose value at the point in question is the blocking probability. In all that follows, we assume that this condition is met and therefore all the differentiation operations we undertake are legitimate.

Note that for the case of Fixed Alternate Routing (FAR) in [35], the shadow price calculation requires the blocking probability of every link in the network. By contrast, in the case of adaptive routing schemes, the calculation of the shadow price of the blocking probability for an OD pair depends on the probability of the states of all the links of the network. This makes the calculation complex and time consuming, specially in the case of asymmetric networks. Let $\underline{\alpha}_{ij}$ denote the vector of the offered traffic for all the states of link (i,j). Then from (4.1), we have

$$\frac{dW(\underline{\lambda},\underline{B})}{d\lambda_{jk}} = \frac{\partial W(\underline{\lambda},\underline{B})}{\partial \lambda_{jk}} + \sum_{[r,s]\in O}\left[\frac{\partial W(\underline{\lambda},\underline{B})}{\partial B_{rs}}\frac{dB_{rs}(\underline{p})}{d\lambda_{jk}}\right], \tag{4.2}$$

$$\frac{\partial W(\underline{\lambda},\underline{B})}{\partial \lambda_{jk}} = w_{jk}\left(1 - B_{jk}(\underline{p})\right), \tag{4.3}$$

$$\frac{dB_{rs}(\underline{p})}{d\lambda_{jk}} = \sum_{(a,b)\in S_{rs}}\sum_{n=0}^{C_{ab}}\frac{\partial B_{rs}(\underline{p})}{\partial p_{ab}(n)}\cdot\frac{dp_{ab}(\underline{\alpha}_{ab},n)}{d\lambda_{jk}}, \tag{4.4}$$

$$\frac{\partial W(\underline{\lambda},\underline{B})}{\partial B_{rs}} = -\lambda_{rs}w_{rs}, \tag{4.5}$$

where $p_{ab}(\underline{\alpha}_{ab},n)$ is the same as $p_{ab}(n)$ defined in (3.1) and (3.2) for LLR, (3.10), (3.11) and (3.12) for ALBA and RALBA, and is written here to explicitly show its dependence on $\underline{\alpha}_{ab}$. $B_{rs}(\underline{p})$ is defined in (3.5) for LLR, (3.15) for ALBA and RALBA and is written here to explicitly show its dependence on $\underline{p}$.

4.2 Shadow Prices for LLR

In (4.4), the partial derivative of B_{rs} with respect to the probability distribution of the links is required. From (3.5) we can obtain

$$\frac{\partial B_{rs}(\underline{p})}{\partial p_{rs}(0)} = \frac{B_{rs}(\underline{p})}{p_{rs}(0)}, \tag{4.6}$$

and for $d \in \mathcal{T}_{rs}$ and $m > T$,

$$\frac{\partial B_{rs}(\underline{p})}{\partial p_{rd}(m)} = -\frac{B_{rs}(\underline{p})\sum_{l=T+1}^{C_{ds}} p_{ds}(\underline{\alpha}_{ds},l)}{1-\left(\sum_{l=T+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd},l)\right)\left(\sum_{l=T+1}^{C_{ds}} p_{ds}(\underline{\alpha}_{ds},l)\right)}. \tag{4.7}$$

Similar derivatives can be obtained for $\frac{\partial B_{rs}(\underline{p})}{\partial p_{ds}(m)}$. The second term in (4.4) can be computed as

$$\frac{dp_{ab}(\underline{\alpha}_{ab}, m)}{d\lambda_{jk}} = \sum_{n=0}^{C_{ab}} \frac{\partial p_{ab}(\underline{\alpha}_{ab}, m)}{\partial \alpha_{ab}(n)} \cdot \frac{d\alpha_{ab}(\underline{\lambda}, \underline{p}, n)}{d\lambda_{jk}}, \tag{4.8}$$

where $\alpha_{ab}(\underline{\lambda}, \underline{p}, n)$ is the same as $\alpha_{ab}(n)$ defined in (3.3) and is written here to show explicitly its dependence on $\underline{\lambda}$ and $\underline{p}$.

We now evaluate each term of (4.8) for every case of n and m. Considering $\prod_{i=l}^{k} \frac{C_{ab}-i+1}{\alpha_{ab}(\underline{\lambda},\underline{p},i)}$ to be 1 for $k < l$ for notational purposes, for the birth-death process given by (3.1), (3.2), (3.3) and (3.4) and for the case $C_{ab} \geq n > m \geq 0$ we get:

$$\begin{aligned} \frac{\partial p_{ab}(\underline{\alpha}_{ab}, m)}{\partial \alpha_{ab}(n)} &= \left[\prod_{i=1}^{m} \frac{C_{ab}-i+1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, i)}\right] \left\{1 + \sum_{k=1}^{C_{ab}} \left[\prod_{i=1}^{k} \frac{C_{ab}-i+1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, i)}\right]\right\}^{-2} \\ &\cdot \frac{1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, n)} \sum_{j=n}^{C_{ab}} \left[\prod_{i=1}^{j} \frac{C_{ab}-i+1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, i)}\right] \\ &= \frac{p_{ab}(\underline{\alpha}_{ab}, m)}{\alpha_{ab}(\underline{\lambda}, \underline{p}, n)} \sum_{j=n}^{C_{ab}} p_{ab}(\underline{\alpha}_{ab}, j). \end{aligned} \tag{4.9}$$

For the case $C_{ab} \geq m \geq n \geq 1$ we get,

$$\begin{aligned} \frac{\partial p_{ab}(\underline{\alpha}_{ab}, m)}{\partial \alpha_{ab}(n)} &= \left[\prod_{i=1}^{m} \frac{C_{ab}-i+1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, i)}\right] \left\{1 + \sum_{k=1}^{C_{ab}} \left[\prod_{i=1}^{k} \frac{C_{ab}-i+1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, i)}\right]\right\}^{-2} \\ &\cdot \frac{1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, n)} \sum_{j=n}^{C_{ab}} \left[\prod_{i=1}^{j} \frac{C_{ab}-i+1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, i)}\right] \\ &- \left\{1 + \sum_{k=1}^{C_{ab}} \left[\prod_{i=1}^{k} \frac{C_{ab}-i+1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, i)}\right]\right\}^{-1} \frac{1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, n)} \prod_{i=1}^{m} \left[\frac{C_{ab}-i+1}{\alpha_{ab}(\underline{\lambda}, \underline{p}, i)}\right] \\ &= \frac{p_{ab}(\underline{\alpha}_{ab}, m)}{\alpha_{ab}(\underline{\lambda}, \underline{p}, n)} \left[\sum_{j=n}^{C_{ab}} p_{ab}(\underline{\alpha}_{ab}, j) - 1\right] \\ &= -\frac{p_{ab}(\underline{\alpha}_{ab}, m)}{\alpha_{ab}(\underline{\lambda}, \underline{p}, n)} \left[\sum_{j=0}^{n-1} p_{ab}(\underline{\alpha}_{ab}, j)\right]. \end{aligned} \tag{4.10}$$

To complete the evaluation of the right hand side of (4.8), we compute the derivative of $\alpha_{ab}(\underline{\lambda}, \underline{p}, n)$ with respect to λ_{jk}. Equations (3.3) and (3.4) describe the dependence of the offered traffic to link (a,b) on the probability distributions of the links. We can write

$$\frac{d\alpha_{ab}(\underline{\lambda},\underline{p},n)}{d\lambda_{jk}} = \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial\lambda_{jk}} + \sum_{(u,v)\in\mathcal{L}}\left[\sum_{i=0}^{C_{uv}} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial p_{uv}(i)}\frac{dp_{uv}(\underline{\alpha}_{uv},i)}{d\lambda_{jk}}\right], \tag{4.11}$$

where $\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial\lambda_{ab}} = 1$, for $1 \le n \le C_{ab}$ and $\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial\lambda_{ab}} = 0$, for $n = 0$. Also for $k \neq b$ from (3.3) we have

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial\lambda_{ak}} = p_{ak}(0)\nu_{ak,b}(n), \quad \text{for} \quad n > T, \tag{4.12}$$

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial\lambda_{ak}} = 0, \quad \text{for} \quad n \le T. \tag{4.13}$$

Similarly for $k \neq a$, i.e., $k \in \mathcal{T}_{ab}$ we have

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial\lambda_{kb}} = p_{kb}(0)\nu_{bk,a}(n), \quad \text{for} \quad n > T, \tag{4.14}$$

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial\lambda_{kb}} = 0, \quad \text{for} \quad n \le T. \tag{4.15}$$

For all other cases, where $(r,s) \notin S_{ab}$, $\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial\lambda_{rs}} = 0$. Now, for $n > T$, consider a link incident to node a, say link (a,v). Then we have

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial p_{av}(0)} = \lambda_{av}\nu_{av,b}(n). \tag{4.16}$$

A similar result holds for a link incident to node b. The following are the equations corresponding to the partial derivatives of the total offered traffic with respect to the probability distributions of the links for LLR from (3.3) and (3.4). In this case, traffic offered to link (r,s), when it is in state n, is considered, i.e., $\alpha_{rs}(\underline{\lambda},\underline{p},n)$, and its partial derivatives are obtained for the probability distributions of the states of those links affecting the offered traffic.

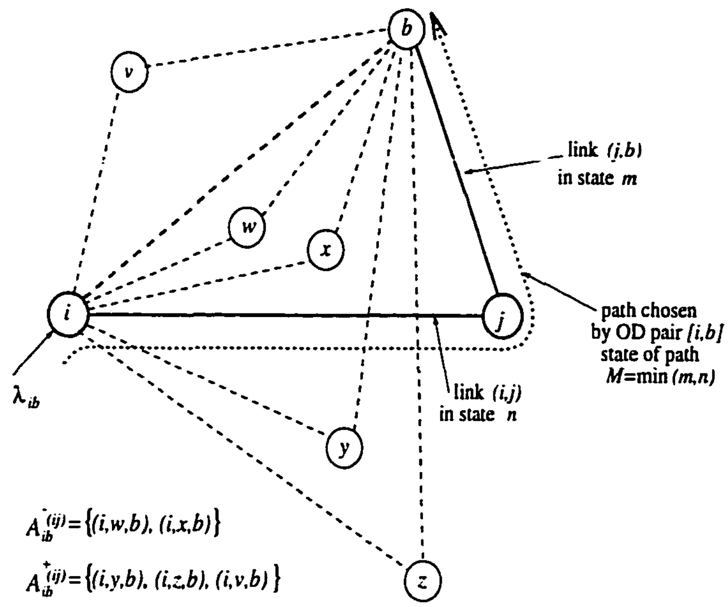

Figure 4.4: Contribution of Offered Traffic to Link (i,j) in state n by OD pair $[i,b]$

As an example consider Figure 4.4 where the traffic is offered to link (i,j), then it can be seen that the links involved in the total offered traffic are links of the form $(q,t) \in S_{ij}$, as link (i,b) in such figure, and those links (u,v) not adjacent to link (i,j) but that satisfy $(u,v) \in S_{ib}$ or $(u,v) \in S_{bj}$, as link (x,b) in such figure. Define $\mathcal{I}_D = 1$ whenever event D occurs and 0 otherwise. Now for the case $n \geq m > T$

$$
\begin{aligned}
\frac{\partial \alpha_{rs}(\underline{\lambda}, \underline{p}, n)}{\partial p_{rv}(m)} &= \lambda_{sv} p_{sv}(\underline{\alpha}_{sv}, 0) \qquad (4.17)\\
&\Bigg\{ \prod_{\substack{d \in T_{sv} \\ d:(s,d),(d,v) \in A^{-}_{sv}(rs)}} \left[1 - \Big(\sum_{i=m}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i) \Big) \Big(\sum_{i=m}^{C_{dv}} p_{dv}(\underline{\alpha}_{dv}, i) \Big) \right] \\
&\cdot \prod_{\substack{d:(s,d),(d,v) \in T_{sv} \\ d \in A^{+}_{sv}(rs)}} \left[1 - \Big(\sum_{i=m+1}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i) \Big) \Big(\sum_{i=m+1}^{C_{dv}} p_{dv}(\underline{\alpha}_{dv}, i) \Big) \right] \Bigg\} \\
&- \sum_{\substack{rb \in S_{rs} \\ b \neq v}} \mathcal{I}_{\{rv \in A^{-}_{rb}(rs)\}} \lambda_{rb} p_{rb}(\underline{\alpha}_{rb}, 0) \\
&\Bigg\{ \sum_{l=T+1}^{m} p_{sb}(\underline{\alpha}_{sb}, l) \sum_{i=l}^{C_{vb}} p_{vb}(\underline{\alpha}_{vb}, i)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^-_{rb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=l}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^+_{rb}(rs)}} \Big[1 - \Big(\sum_{i=l+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big] \\
&+ \Big(\sum_{\substack{l=n+1 \\ m=n}}^{C_{sb}} p_{sb}(\underline{\alpha}_{sb}, l)\Big)\Big(\sum_{i=n}^{C_{vb}} p_{vb}(\underline{\alpha}_{vb}, i)\Big) \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^-_{rb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=n}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=n}^{C_{db}} p_{db}(i)\Big)\Big] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^+_{rb}(rs)}} \Big[1 - \Big(\sum_{i=n+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=n+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big]\Big\} \\
&- \sum_{\substack{rb \in S_{rs} \\ b \neq v}} \mathcal{I}_{\{rv \in A^+_{rb}(rs),\ m > T+1\}} \lambda_{rb} p_{rb}(\underline{\alpha}_{rb}, 0) \\
&\Big\{ \sum_{l=T+1}^{m-1} p_{sb}(\underline{\alpha}_{sb}, l) \sum_{i=l+1}^{C_{vb}} p_{vb}(\underline{\alpha}_{vb}, i) \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^-_{rb}(rs)}} \Big[1 - \Big(\sum_{i=l}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^+_{rb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=l+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big]\Big\}.
\end{aligned}
$$

For the case $m > n > T$

$$
\begin{aligned}
\frac{\partial \alpha_{rs}(\underline{\lambda}, \underline{p}, n)}{\partial p_{rv}(m)} &= \lambda_{sv} p_{sv}(\underline{\alpha}_{sv}, 0) \\
&\Big\{ \prod_{\substack{d \in \mathcal{T}_{sv} \\ d:(s,d),(d,v) \in A^-_{sv}(rs)}} \Big[1 - \Big(\sum_{i=n}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i)\Big)\Big(\sum_{i=n}^{C_{dv}} p_{dv}(\underline{\alpha}_{dv}, i)\Big)\Big] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{sv} \\ d:(s,d),(d,v) \in A^+_{sv}(rs)}} \Big[1 - \Big(\sum_{i=n+1}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i)\Big)\Big(\sum_{i=n+1}^{C_{dv}} p_{dv}(\underline{\alpha}_{dv}, i)\Big)\Big]\Big\} \\
&- \sum_{\substack{rb \in S_{rs} \\ b \neq v}} \mathcal{I}_{\{rv \in A^-_{rb}(rs)\}} \lambda_{rb} p_{rb}(\underline{\alpha}_{rb}, 0)
\end{aligned}
\tag{4.18}
$$

$$
\begin{aligned}
&\Bigg\{ \sum_{l=T+1}^{n} p_{sb}(\underline{\alpha}_{sb}, l) \sum_{i=l}^{C_{vb}} p_{vb}(\underline{\alpha}_{vb}, i) \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^-_{rb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=l}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i) \Big) \Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i) \Big) \Big] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^+_{rb}(rs)}} \Big[1 - \Big(\sum_{i=l+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i) \Big) \Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i) \Big) \Big] \\
&+ \Big(\sum_{l=n+1}^{C_{sb}} p_{sb}(\underline{\alpha}_{sb}, l) \Big) \Big(\sum_{i=n}^{C_{vb}} p_{vb}(\underline{\alpha}_{vb}, i) \Big) \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^-_{rb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=n}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i) \Big) \Big(\sum_{i=n}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i) \Big) \Big] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^+_{rb}(rs)}} \Big[1 - \Big(\sum_{i=n+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i) \Big) \Big(\sum_{i=n+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i) \Big) \Big] \Bigg\} \\
&- \sum_{\substack{rb \in S_{rs} \\ b \neq v}} \mathcal{I}_{\{rv \in A^+_{rb}(rs)\}} \lambda_{rb} p_{rb}(\underline{\alpha}_{rb}, 0) \\
&\Bigg\{ \sum_{l=T+1}^{n} p_{sb}(\underline{\alpha}_{sb}, l) \sum_{i=l+1}^{C_{vb}} p_{vb}(\underline{\alpha}_{vb}, i) \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^-_{rb}(rs)}} \Big[1 - \Big(\sum_{i=l}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i) \Big) \Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i) \Big) \Big] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^+_{rb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=l+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i) \Big) \Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i) \Big) \Big] \\
&+ \Big(\sum_{l=n+1}^{C_{sb}} p_{sb}(\underline{\alpha}_{sb}, l) \Big) \Big(\sum_{i=n+1}^{C_{vb}} p_{vb}(\underline{\alpha}_{vb}, i) \Big) \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^-_{rb}(rs)}} \Big[1 - \Big(\sum_{i=n}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i) \Big) \Big(\sum_{i=n}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i) \Big) \Big] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^+_{rb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=n+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i) \Big) \Big(\sum_{i=n+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i) \Big) \Big] \Bigg\}.
\end{aligned}
$$

Similarly for $\frac{\partial \alpha_{rs}(\underline{\lambda},\underline{p},n)}{\partial p_{vs}(m)}$. The following is the case where the link with respect to which we get the partial derivative is not incident to any of the nodes r or s, but it forms part of a two-link alternate route of an OD pair adjacent to OD pair $[r,s]$, then for $m \geq n > T$

$$
\begin{aligned}
\frac{\partial \alpha_{rs}(\underline{\lambda},\underline{p},n)}{\partial p_{vb}(m)} = & -\lambda_{rb} p_{rb}(\underline{\alpha}_{rb},0)\mathcal{I}_{\{vb\in A^-_{rb}(rs)\}} \qquad (4.19) \\
& \left\{ \sum_{l=T+1}^{m} p_{sb}(\underline{\alpha}_{sb},l) \sum_{i=l}^{C_{rv}} p_{rv}(\underline{\alpha}_{rv},i) \right. \\
& \cdot \prod_{\substack{d\in\mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^-_{rb}(rs) \\ d\neq v}} \left[1-\Big(\sum_{i=l}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd},i)\Big)\Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\right] \\
& \cdot \prod_{\substack{d\in\mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^+_{rb}(rs)}} \left[1-\Big(\sum_{i=l+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd},i)\Big)\Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\right] \\
& + \Big(\sum_{l=n+1}^{C_{sb}} p_{sb}(\underline{\alpha}_{sb},l)\mathcal{I}_{\{m=n\}}\Big)\Big(\sum_{i=n}^{C_{rv}} p_{rv}(\underline{\alpha}_{rv},i)\Big) \\
& \cdot \prod_{\substack{d\in\mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^-_{rb}(rs) \\ d\neq v}} \left[1-\Big(\sum_{i=n}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd},i)\Big)\Big(\sum_{i=n}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\right] \\
& \left. \cdot \prod_{\substack{d\in\mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^+_{rb}(rs)}} \Big(\sum_{i=n+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd},i)\Big)\Big(\sum_{i=n+1}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\Big]\right\} \\
& - \lambda_{rb} p_{rb}(\underline{\alpha}_{rb},0)\mathcal{I}_{\{vb\in A^+_{rb}(rs),m>T+1\}} \\
& \left\{ \sum_{l=T+1}^{m-1} p_{sb}(\underline{\alpha}_{sb},l) \sum_{i=l+1}^{C_{rv}} p_{rv}(\underline{\alpha}_{rv},i) \right. \\
& \cdot \prod_{\substack{d\in\mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^-_{rb}(rs)}} \left[1-\Big(\sum_{i=l}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd},i)\Big)\Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\right] \\
& \left. \cdot \prod_{\substack{d\in\mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^+_{rb}(rs) \\ d\neq v}} \left[1-\Big(\sum_{i=l+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd},i)\Big)\Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\right]\right\} \\
& - \lambda_{bs} p_{bs}(\underline{\alpha}_{bs},0)\mathcal{I}_{\{vb\in A^-_{sb}(rs)\}}
\end{aligned}
$$

$$
\begin{aligned}
& \Big\{ \sum_{l=T+1}^{m} p_{rb}(\underline{\alpha}_{rb}, l) \sum_{i=l}^{C_{sv}} p_{sv}(\underline{\alpha}_{sv}, i) \\
& \cdot \prod_{\substack{d \in \mathcal{T}_{sb} \\ d:(r,d),(d,b) \in A^-_{sb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=l}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i)\Big)\Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big] \\
& \cdot \prod_{\substack{d \in \mathcal{T}_{sb} \\ d:(r,d),(d,b) \in A^+_{sb}(rs)}} \Big[1 - \Big(\sum_{i=l+1}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i)\Big)\Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big] \\
& + \Big(\sum_{l=n+1}^{C_{rb}} p_{rb}(\underline{\alpha}_{rb}, l)\mathcal{I}_{\{m=n\}}\Big)\Big(\sum_{i=n}^{C_{sv}} p_{sv}(\underline{\alpha}_{sv}, i)\Big) \\
& \cdot \prod_{\substack{d \in \mathcal{T}_{sb} \\ d:(r,d),(d,b) \in A^-_{sb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=n}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i)\Big)\Big(\sum_{i=n}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big] \\
& \cdot \prod_{\substack{d \in \mathcal{T}_{sb} \\ d:(r,d),(d,b) \in A^+_{sb}(rs)}} \Big[1 - \Big(\sum_{i=n+1}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i)\Big)\Big(\sum_{i=n+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big]\Big\} \\
& - \lambda_{sb} p_{sb}(\underline{\alpha}_{sb}, 0)\mathcal{I}_{\{vb \in A^+_{sb}(rs), m > T+1\}} \\
& \Big\{ \sum_{l=T+1}^{m-1} p_{rb}(\underline{\alpha}_{rb}, l) \sum_{i=l+1}^{C_{sv}} p_{sv}(\underline{\alpha}_{sv}, i) \\
& \cdot \prod_{\substack{d \in \mathcal{T}_{sb} \\ d:(r,d),(d,b) \in A^-_{sb}(rs)}} \Big[1 - \Big(\sum_{i=l}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i)\Big)\Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big] \\
& \cdot \prod_{\substack{d \in \mathcal{T}_{sb} \\ d:(r,d),(d,b) \in A^+_{sb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=l+1}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i)\Big)\Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big]\Big\}.
\end{aligned}
$$

For the case $m > n > T$,

$$
\begin{aligned}
\frac{\partial \alpha_{rs}(\underline{\lambda}, \underline{p}, n)}{\partial p_{vb}(m)} = & -\lambda_{rb} p_{rb}(\underline{\alpha}_{rb}, 0)\mathcal{I}_{\{vb \in A^-_{rb}(rs)\}} \qquad (4.20) \\
& \Big\{ \sum_{l=T+1}^{n} p_{sb}(\underline{\alpha}_{sb}, l) \sum_{i=l}^{C_{rv}} p_{rv}(\underline{\alpha}_{rv}, i) \\
& \cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b) \in A^-_{rb}(rs) \\ d \neq v}} \Big[1 - \Big(\sum_{i=l}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^{+}_{rb}(rs)}} \left[1 - \Big(\sum_{i=l+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\right] \\
&+ \Big(\sum_{l=n+1}^{C_{sb}} p_{sb}(\underline{\alpha}_{sb}, l)\Big)\Big(\sum_{i=n}^{C_{vb}} p_{vb}(\underline{\alpha}_{vb}, i)\Big) \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^{-}_{rb}(rs) \\ d \neq v}} \left[1 - \Big(\sum_{i=n}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=n}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\right] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^{+}_{rb}(rs)}} \left[1 - \Big(\sum_{i=n+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=n+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\right]\Bigg\} \\
&- \lambda_{rb} p_{rb}(\underline{\alpha}_{rb}, 0)\mathcal{I}_{\{vb \in A^{+}_{rb}(rs)\}} \\
&\Bigg\{ \sum_{l=T+1}^{n} p_{sb}(\underline{\alpha}_{sb}, l) \sum_{i=l+1}^{C_{rv}} p_{rv}(\underline{\alpha}_{rv}, i) \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^{-}_{rb}(rs)}} \left[1 - \Big(\sum_{i=l}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\right] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^{+}_{rb}(rs) \\ d \neq v}} \left[1 - \Big(\sum_{i=l+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\right] \\
&+ \Big(\sum_{l=n+1}^{C_{sb}} p_{sb}(\underline{\alpha}_{sb}, l)\Big)\Big(\sum_{i=n+1}^{C_{rv}} p_{rv}(\underline{\alpha}_{rv}, i)\Big) \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^{-}_{rb}(rs)}} \left[1 - \Big(\sum_{i=n}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=n}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\right] \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{rb} \\ d:(r,d),(d,b)\in A^{+}_{rb}(rs) \\ d \neq v}} \left[1 - \Big(\sum_{i=n+1}^{C_{rd}} p_{rd}(\underline{\alpha}_{rd}, i)\Big)\Big(\sum_{i=n+1}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\right]\Bigg\} \\
&- \lambda_{sb} p_{sb}(\underline{\alpha}_{sb}, 0)\mathcal{I}_{\{vb \in A^{-}_{sb}(rs)\}} \\
&\Bigg\{ \sum_{l=T+1}^{n} p_{rb}(\underline{\alpha}_{rb}, l) \sum_{i=l}^{C_{sv}} p_{sv}(\underline{\alpha}_{sv}, i) \\
&\cdot \prod_{\substack{d \in \mathcal{T}_{sb} \\ d:(r,d),(d,b)\in A^{-}_{sb}(rs) \\ d \neq v}} \left[1 - \Big(\sum_{i=l}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd}, i)\Big)\Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db}, i)\Big)\right]
\end{aligned}
$$

$$
\begin{aligned}
\cdot & \prod_{\substack{d\in\mathcal{T}_{sb}\\ d:(r,d),(d,b)\in A^{+}_{sb}(rs)}} \Big[1-\Big(\sum_{i=l+1}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd},i)\Big)\Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\Big] \\
+ & \Big(\sum_{l=n+1}^{C_{rb}} p_{rb}(\underline{\alpha}_{rb},l)\Big)\Big(\sum_{i=n}^{C_{sv}} p_{sv}(\underline{\alpha}_{sv},i)\Big) \\
\cdot & \prod_{\substack{d\in\mathcal{T}_{sb}\\ d:(r,d),(d,b)\in A^{-}_{sb}(rs)\\ d\neq v}} \Big[1-\Big(\sum_{i=n}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd},i)\Big)\Big(\sum_{i=n}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\Big] \\
\cdot & \prod_{\substack{d\in\mathcal{T}_{sb}\\ d:(r,d),(d,b)\in A^{+}_{sb}(rs)}} \Big[1-\Big(\sum_{i=n+1}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd},i)\Big)\Big(\sum_{i=n+1}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\Big]\Big\} \\
- & \lambda_{sb} p_{sb}(\underline{\alpha}_{sb},0)\mathcal{I}_{\{vb\in A^{+}_{sb}(rs)\}} \\
\Big\{ & \sum_{l=T+1}^{n} p_{rb}(\underline{\alpha}_{rb},l) \sum_{i=l+1}^{C_{sv}} p_{sv}(\underline{\alpha}_{sv},i) \\
\cdot & \prod_{\substack{d\in\mathcal{T}_{sb}\\ d:(r,d),(d,b)\in A^{-}_{sb}(rs)}} \Big[1-\Big(\sum_{i=l}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd},i)\Big)\Big(\sum_{i=l}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\Big] \\
\cdot & \prod_{\substack{d\in\mathcal{T}_{sb}\\ d:(r,d),(d,b)\in A^{+}_{sb}(rs)\\ d\neq v}} \Big[1-\Big(\sum_{i=l+1}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd},i)\Big)\Big(\sum_{i=l+1}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\Big] \\
+ & \Big(\sum_{l=n+1}^{C_{rb}} p_{rb}(\underline{\alpha}_{rb},l)\Big)\Big(\sum_{i=n+1}^{C_{sv}} p_{sv}(\underline{\alpha}_{sv},i)\Big) \\
\cdot & \prod_{\substack{d\in\mathcal{T}_{sb}\\ d:(r,d),(d,b)\in A^{-}_{sb}(rs)}} \Big[1-\Big(\sum_{i=n}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd},i)\Big)\Big(\sum_{i=n}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\Big] \\
\cdot & \prod_{\substack{d\in\mathcal{T}_{sb}\\ d:(r,d),(d,b)\in A^{+}_{sb}(rs)\\ d\neq v}} \Big[1-\Big(\sum_{i=n+1}^{C_{sd}} p_{sd}(\underline{\alpha}_{sd},i)\Big)\Big(\sum_{i=n+1}^{C_{db}} p_{db}(\underline{\alpha}_{db},i)\Big)\Big]\Big\}.
\end{aligned}
$$

Substituting (4.11) in (4.8) we get

$$
\begin{aligned}
\frac{dp_{ab}(\underline{\alpha}_{ab},m)}{d\lambda_{jk}} &= \sum_{n=0}^{C_{ab}} \frac{\partial p_{ab}(\underline{\alpha}_{ab},m)}{\partial \alpha_{ab}(n)} \frac{\partial \alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial \lambda_{jk}} \\
&+ \sum_{n=0}^{C_{ab}} \frac{\partial p_{ab}(\underline{\alpha}_{ab},m)}{\partial \alpha_{ab}(n)} \Big\{ \sum_{(u,v)\in\mathcal{L}} \sum_{i=0}^{C_{uv}} \frac{\partial \alpha_{ab}(\underline{\lambda},\underline{p},n)}{\partial p_{uv}(i)} \frac{dp_{uv}(\underline{\alpha}_{uv},i)}{d\lambda_{jk}} \Big\}. \qquad (4.21)
\end{aligned}
$$

Equations (4.21) are a set of simultaneous linear equations for $\frac{dp_{ab}(\underline{\alpha}_{ab},m)}{d\lambda_{jk}}$. By the earlier assumptions, the derivatives $\frac{dp_{ab}(\underline{\alpha}_{ab},m)}{d\lambda_{jk}}$ exist and so the set of equations (4.21) has a solution. We solve (4.21) and use this solution to evaluate $\frac{dB_{rs}(\underline{p})}{d\lambda_{jk}}$ in (4.4). Then (4.2) can be used to evaluate $\frac{dW(\underline{\lambda},\underline{B})}{d\lambda_{jk}}$. Results on shadow prices can be found at the end of the next section where LLR and ALBA are compared.

4.3 Shadow Prices for ALBA

To compute the shadow prices of ALBA, we adopt the same approach as for LLR. We first calculate the partial derivative of B_{uv} with respect to the probability distribution for the states of the links. From (3.15), we have

$$\frac{\partial B_{uv}(\underline{p})}{\partial p_{uv}(C_{uv})} = \frac{B_{uv}(\underline{p})}{p_{uv}(C_{uv})}, \tag{4.22}$$

and for $b \in \mathcal{T}_{uv}$, from (3.7) and (3.15) we get

$$\frac{\partial B_{uv}(\underline{p})}{\partial p_{ub}(m)} = \begin{cases} \frac{B_{uv}(\underline{p})(1-P_{bv}(K-1))}{[1-(1-P_{ub}(K-1))(1-P_{bv}(K-1))]}, & \forall m \in \mathcal{A}^{ub}_{K-1} \\ 0, & \text{otherwise.} \end{cases} \tag{4.23}$$

An equation similar to (4.23) holds for $\frac{\partial B_{uv}(\underline{p})}{\partial p_{bv}(m)}$ for $b \in \mathcal{T}_{uv}$. For the following expressions we use $\underline{\Psi}$ as the matrix whose (i,j)th element is determined by (3.9).

Consider the second term in (4.4) which can be computed as

$$\frac{dp_{rs}(\underline{\alpha}_{rs},m)}{d\lambda_{jk}} = \sum_{n=0}^{K-1} \frac{\partial p_{rs}(\underline{\alpha}_{rs},m)}{\partial \alpha_{rs}(n)} \cdot \frac{d\alpha_{rs}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{d\lambda_{jk}}, \tag{4.24}$$

where, in order to indicate the dependence of the offered traffic to a link (r,s), we write it as $\alpha_{rs}(\underline{\lambda},\underline{p},\underline{\Psi},n)$. The first term in (4.24) is computed using (3.10) to (3.12) as follows:

$$\frac{\partial p_{ij}(\underline{\alpha}_{ij},0)}{\partial \alpha_{ij}(n)} = -p_{ij}(\underline{\alpha}_{ij},0) \left\{ \mathcal{I}_{\{n=0\}} \sum_{x=0}^{C_{ij}-r_1^{ij}-1} p_{ij}(\underline{\alpha}_{ij},x) \right. \tag{4.25}$$

$$+\frac{\mathcal{I}_{\{n>0\}}}{\alpha_{ij}(\underline{\lambda},\underline{p},\underline{\Psi},n)}\sum_{x=C_{ij}-r_n^{ij}+1}^{C_{ij}-r_{n+1}^{ij}}(x-C_{ij}+r_n^{ij})p_{ij}(\underline{\alpha}_{ij},x)$$

$$\left.+\frac{r_n^{ij}-r_{n+1}^{ij}}{\alpha_{ij}(\underline{\lambda},\underline{p},\underline{\Psi},n)}\sum_{u=n+1}^{K-1}\sum_{x=C_{ij}-r_u^{ij}+1}^{C_{ij}-r_{u+1}^{ij}}p_{ij}(\underline{\alpha}_{ij},x)\right\},$$

where $\mathcal{I}_{\{n=0\}}=1$, when $n=0$ and 0 otherwise, and $\mathcal{I}_{\{n>0\}}=1$, when $n>0$ and 0 otherwise. Let $m\in\mathcal{A}_w^{ij}$, i.e., $C_{ij}-r_w^{ij}\le m<C_{ij}-r_{w+1}^{ij}$. Then for the case $w<n$,

$$\frac{\partial p_{ij}(\underline{\alpha}_{ij},m)}{\partial\alpha_{ij}(n)}=\frac{p_{ij}(\underline{\alpha}_{ij},m)}{p_{ij}(\underline{\alpha}_{ij},0)}\cdot\frac{\partial p_{ij}(\underline{\alpha}_{ij},0)}{\partial\alpha_{ij}(n)}. \tag{4.26}$$

For the case $n\le w$ and $m>0$ we get from (3.10), (3.11) and (3.12) that

$$\begin{aligned}\frac{\partial p_{ij}(\underline{\alpha}_{ij},m)}{\partial\alpha_{ij}(n)} &= p_{ij}(\underline{\alpha}_{ij},m-1)\\ &+\frac{p_{ij}(\underline{\alpha}_{ij},m)}{p_{ij}(\underline{\alpha}_{ij},0)}\cdot\frac{\partial p_{ij}(\underline{\alpha}_{ij},0)}{\partial\alpha_{ij}(n)},\qquad 0<m\le C_{ij}-r_1^{ij},\end{aligned} \tag{4.27}$$

$$\begin{aligned}\frac{\partial p_{ij}(\underline{\alpha}_{ij},m)}{\partial\alpha_{ij}(n)} &= \frac{m-C_{ij}+r_n^{ij}}{\alpha_{ij}(\underline{\lambda},\underline{p},\underline{\Psi},n)}p_{ij}(\underline{\alpha}_{ij},m)\\ &+\frac{p_{ij}(\underline{\alpha}_{ij},m)}{p_{ij}(\underline{\alpha}_{ij},0)}\cdot\frac{\partial p_{ij}(\underline{\alpha}_{ij},0)}{\partial\alpha_{ij}(n)},\quad n>0,\quad C_{ij}-r_n^{ij}<m\le C_{ij}-r_{n+1}^{ij},\end{aligned} \tag{4.28}$$

$$\begin{aligned}\frac{\partial p_{ij}(\underline{\alpha}_{ij},m)}{\partial\alpha_{ij}(n)} &= \frac{r_n^{ij}-r_{n+1}^{ij}}{\alpha_{ij}(\underline{\lambda},\underline{p},\underline{\Psi},n)}p_{ij}(\underline{\alpha}_{ij},m)\\ &+\frac{p_{ij}(\underline{\alpha}_{ij},m)}{p_{ij}(\underline{\alpha}_{ij},0)}\cdot\frac{\partial p_{ij}(\underline{\alpha}_{ij},0)}{\partial\alpha_{ij}(n)},\quad C_{ij}-r_{n+1}^{ij}<m.\end{aligned} \tag{4.29}$$

To complete the evaluation of the right hand side of (4.24) we compute the derivative of $\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)$ with respect to λ_{ij}. From (3.13) and (3.14) it can be seen that the offered traffic to link (a,b) depends on the probability distributions for the aggregate states of those links adjacent to link (a,b) and on $\underline{\Psi}$ for those same adjacent links. Hence we can write

$$\frac{d\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{d\lambda_{ij}}=\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial\lambda_{ij}}$$

$$+ \sum_{w:[a,w]\in\mathcal{O}} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial p_{aw}(C_{aw})}\frac{dp_{aw}(C_{aw})}{d\lambda_{ij}}$$
$$+ \sum_{w:[w,b]\in\mathcal{O}} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial p_{wb}(C_{wb})}\frac{dp_{wb}(C_{wb})}{d\lambda_{ij}}$$
$$+ \sum_{w:[a,w]\in\mathcal{O}} \sum_{H=0}^{K-1} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial P_{aw}(H)}\frac{dP_{aw}(H)}{d\lambda_{ij}}$$
$$+ \sum_{w:[w,b]\in\mathcal{O}} \sum_{H=0}^{K-1} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial P_{wb}(H)}\frac{dP_{wb}(H)}{d\lambda_{ij}}$$
$$+ \sum_{w:[a,w]\in\mathcal{O}} \sum_{\substack{d\in\mathcal{T}_{aw},\\ d\neq b}} \sum_{G=n}^{K-2} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial\Psi^d_{aw}(G)}\frac{d\Psi^d_{aw}(\underline{p},G)}{d\lambda_{ij}}$$
$$+ \sum_{w:[wb]\in\mathcal{O}} \sum_{\substack{d\in\mathcal{T}_{wb},\\ d\neq a}} \sum_{G=n}^{K-2} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial\Psi^d_{wb}(G)}\frac{d\Psi^d_{wb}(\underline{p},G)}{d\lambda_{ij}}. \qquad (4.30)$$

Now from equations (3.13) and (3.14) we can get the partial derivatives needed in (4.30) of the offered traffic, $\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)$, to the links with respect to the OD pair external arrival rates, λ_{ij}, and the probability distributions of the links, $p_{ae}(\underline{\alpha}_{ae},n)$, as follows

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial\lambda_{ij}} = 1, \qquad if \quad (a,b)=(i,j), \quad \forall n, \qquad (4.31)$$

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial\lambda_{ij}} = p_{ij}(C_{ij})\nu_{ij,b}(n), \qquad 0\leq n<K-1, \quad (i,j)\in S_{ab}, \qquad (4.32)$$

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial p_{ae}(C_{ae})} = \lambda_{ae}\nu_{ae,b}(n), \qquad n<K-1, \qquad (4.33)$$

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial p_{eb}(C_{eb})} = \lambda_{eb}\nu_{be,a}(n), \qquad n<K-1. \qquad (4.34)$$

For the calculation of (4.30) we also need the partial derivatives of $\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)$ with respect to P_{aw} and P_{wb}, which are obtained as follows:

$$
\frac{\partial \alpha_{ab}(\underline{\lambda}, \underline{p}, \underline{\Psi}, n)}{\partial P_{aw}(H)} = \begin{cases} \lambda_{wb} p_{wb}(C_{wb}) \prod\limits_{\substack{d:(w,d),(d,b)\in A^{-}_{wb}(ab),\\ d\neq a}} \Psi^{d}_{wb}(n+1)\cdot \\ \cdot \prod\limits_{\substack{d:(w,d),(d,b)\in A^{+}_{wb}(ab),\\ d\neq a}} \Psi^{d}_{wb}(n), & H \leq n, \\ \lambda_{wb} p_{wb}(C_{wb}) \prod\limits_{\substack{d:(w,d),(d,b)\in A^{-}_{wb}(ab),\\ d\neq a}} \Psi^{d}_{wb}(H+1)\cdot \\ \cdot \prod\limits_{\substack{d:(w,d),(d,b)\in A^{+}_{wb}(ab),\\ d\neq a}} \Psi^{d}_{wb}(H), & n < H, \end{cases} \tag{4.35}
$$

and similarly for P_{wb}. The following are the remaining partial derivatives needed to compute (4.30) for ALBA

$$
\frac{dP_{aw}(H)}{d\lambda_{ij}} = \sum_{m\in A^{aw}_{H}} \frac{\partial P_{aw}(H)}{\partial p_{aw}(m)} \cdot \frac{dp_{aw}(\underline{\alpha}_{aw}, m)}{d\lambda_{ij}}, \tag{4.36}
$$

where

$$
\frac{\partial P_{aw}(H)}{\partial p_{aw}(m)} = \begin{cases} 1, & C_{aw} - r^{aw}_{H} \leq m < C_{aw} - r^{aw}_{H+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{4.37}
$$

$$
\begin{aligned} \frac{d\Psi^{d}_{aw}(\underline{p}, G)}{d\lambda_{ij}} &= \sum_{m=0}^{C_{ad}} \frac{\partial \Psi^{d}_{aw}(\underline{p}, G)}{\partial p_{ad}(m)} \frac{dp_{ad}(\underline{\alpha}_{ad}, m)}{d\lambda_{ij}} \\ &+ \sum_{m=0}^{C_{dw}} \frac{\partial \Psi^{d}_{aw}(\underline{p}, G)}{\partial p_{dw}(m)} \frac{dp_{dw}(\underline{\alpha}_{dw}, m)}{d\lambda_{ij}}, \end{aligned} \tag{4.38}
$$

$$
\frac{\partial \Psi^{d}_{uw}(\underline{p}, G)}{\partial p_{ud}(m)} = \begin{cases} -\sum\limits_{x=0}^{C_{dw}-r^{dw}_{G}-1} p_{dw}(\underline{\alpha}_{dw}, x), & 0 < G < K, \quad m < C_{ud} - r^{ud}_{G}; \\ 0, & G = 0, \quad \forall m. \end{cases} \tag{4.39}
$$

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial\Psi^{d}_{aw}(G)} = \begin{cases} \lambda_{aw}p_{aw}(C_{aw})\sum_{H=0}^{n}P_{wb}(H)\prod_{\substack{z\in A^{-}_{aw}(ab),\\ z\neq b,d}}\Psi^{z}_{aw}(n+1) & \\ \cdot\prod_{\substack{z\in A^{+}_{aw}(ab),\\ z\neq b}}\Psi^{z}_{aw}(n), \qquad d\in A^{-}_{aw}(ab), & G=n+1; \\ \lambda_{aw}p_{aw}(C_{aw})P_{wb}(n+1)\prod_{\substack{z\in A^{-}_{aw}(ab),\\ z\neq b}}\Psi^{z}_{aw}(n+2) & \\ \cdot\prod_{\substack{z\in A^{+}_{aw}(ab),\\ z\neq b,d}}\Psi^{z}_{aw}(n+1), \qquad d\in A^{+}_{aw}(ab), & G=n+1; \\ \lambda_{aw}p_{aw}(C_{aw})\sum_{H=0}^{n}P_{wb}(H)\prod_{\substack{z\in A^{-}_{aw}(ab),\\ z\neq b}}\Psi^{z}_{aw}(n+1) & \\ \cdot\prod_{\substack{z\in A^{+}_{aw}(ab),\\ z\neq b,d}}\Psi^{z}_{aw}(n), \qquad d\in A^{+}_{aw}(ab), & G=n; \end{cases} \tag{4.40}$$

and for the case $G > n+1$ we have

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial\Psi^{d}_{aw}(G)} = \begin{cases} \lambda_{aw}p_{aw}(C_{aw})P_{wb}(G-1)\prod_{\substack{z\in A^{-}_{aw}(ab),\\ z\neq b,d}}\Psi^{z}_{aw}(G) & \\ \cdot\prod_{\substack{z\in A^{+}_{aw}(ab),\\ z\neq b}}\Psi^{z}_{aw}(G-1), \qquad d\in A^{-}_{aw}(ab), & G>n+1; \\ \lambda_{aw}p_{aw}(C_{aw})P_{wb}(G)\prod_{\substack{z\in A^{-}_{aw}(ab),\\ z\neq b}}\Psi^{z}_{aw}(G+1) & \\ \cdot\prod_{\substack{z\in A^{+}_{aw}(ab),\\ z\neq b,d}}\Psi^{z}_{aw}(G), \qquad d\in A^{+}_{aw}(ab), & G>n+1, \end{cases} \tag{4.41}$$

Next, for $\Psi^{d}_{wb}(G)$, where $G = n$, we get

$$\begin{aligned} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Psi},n)}{\partial\Psi^{d}_{wb}(G)} &= \lambda_{wb}p_{wb}(C_{wb})\sum_{H=0}^{n}P_{aw}(H) \qquad (4.42)\\ &\cdot\prod_{\substack{z\in A^{-}_{wb}(ab),\\ z\neq a}}\Psi^{z}_{wb}(n+1)\cdot\prod_{\substack{z\in A^{+}_{wb}(ab),\\ z\neq a,d}}\Psi^{z}_{wb}(n),\;\; d\in A^{+}_{wb}(ab),\;\; G=n; \end{aligned}$$

and for $G = n+1$ we have that this partial derivative is obtained by the following expression

$$\frac{\partial \alpha_{ab}(\underline{\lambda}, \underline{p}, \underline{\Psi}, n)}{\partial \Psi^d_{wb}(G)} = \begin{cases} \lambda_{wb} p_{wb}(C_{wb}) \sum_{H=0}^{n} P_{aw}(H) \prod_{\substack{z \in A^-_{wb}(ab), \\ z \neq a,d}} \Psi^z_{wb}(n+1) \\ \cdot \prod_{\substack{z \in A^+_{wb}(ab), \\ z \neq a}} \Psi^z_{wb}(n), \qquad d \in A^-_{wb}(ab), \qquad G = n+1; \\ \lambda_{wb} p_{wb}(C_{wb}) P_{aw}(n+1) \prod_{\substack{z \in A^-_{wb}(ab), \\ z \neq a}} \Psi^z_{wb}(n+2) \\ \cdot \prod_{\substack{z \in A^+_{wb}(ab), \\ z \neq a,d}} \Psi^z_{wb}(n+1), \qquad d \in A^+_{wb}(ab), \qquad G = n+1; \end{cases} \tag{4.43}$$

and for the case $G > n+1$ we have

$$\frac{\partial \alpha_{ab}(\underline{\lambda}, \underline{p}, \underline{\Psi}, n)}{\partial \Psi^d_{wb}(G)} = \begin{cases} \lambda_{wb} p_{wb}(C_{wb}) P_{aw}(G-1) \prod_{\substack{z \in A^-_{wb}(ab), \\ z \neq a,d}} \Psi^z_{wb}(G) \\ \cdot \prod_{\substack{z \in A^+_{wb}(ab), \\ z \neq a}} \Psi^z_{wb}(G-1), \qquad d \in A^-_{wb}(ab), \qquad G > n+1; \\ \lambda_{wb} p_{wb}(C_{wb}) P_{aw}(G) \prod_{\substack{z \in A^-_{wb}(ab), \\ z \neq a}} \Psi^z_{wb}(G+1) \\ \cdot \prod_{\substack{z \in A^+_{wb}(ab), \\ z \neq a,d}} \Psi^z_{wb}(G), \qquad d \in A^+_{wb}(ab), \qquad G > n+1, \end{cases} \tag{4.44}$$

Substituting (4.31) - (4.39) into (4.30), and then substituting the result into (4.24), we get a set of simultaneous linear equations for $\frac{dp_{ab}(\underline{\alpha}_{ab}, m)}{d\lambda ij}$. We solve this and use the solution to evaluate $\frac{dB_{rs}(p)}{d\lambda_{jk}}$ in (4.4). Then (4.2) can be used to evaluate $\frac{dW(\lambda, \underline{B})}{d\lambda_{jk}}$, the shadow price.

In order to study the interactions of exogenous traffic and mismatched capacity on the blocking probability and the network rate of return, a number of numerical examples were conducted. In all the numerical examples, blocking probability is used as the performance measure (all the w_{ij}'s are chosen to be 1). Since the calculation of the shadow price of the blocking probability for an OD pair depends

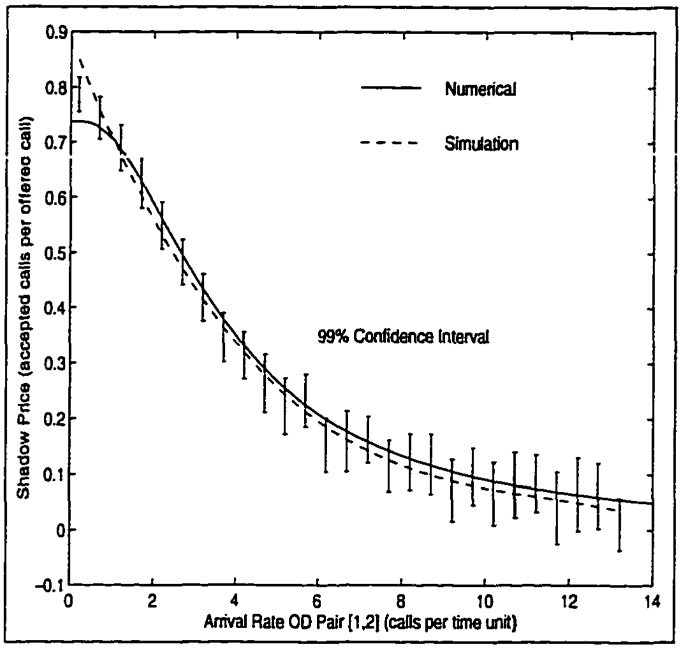

Figure 4.5: Simulated and Numerical Shadow Prices for the Four-Node Network using LLR with $T = 2$

on the probability of the states of all the links of the network, the calculations are complex and time consuming especially in the case of asymmetric networks and so we restrict attention to small networks, with 4 and 5 nodes as those in figures 3.1 and 3.6. For large sized asymmetric networks, further approximations along the lines of the work in [44] and [61] need to be done.

To verify the accuracy of our numerical calculation of the shadow prices, we compared the shadow prices obtained from numerical calculation with those obtained from simulation for the 4-node network of Figure 3.1 and 5-node network of Figure 3.6, using LLR with trunk reservation $T = 2$, as a function of the external arrival rate of OD pair [1, 2].

In the simulations, in order to evaluate the derivative of the network rate of return, W, with respect to one of the external arrival rates, say λ_{jk}, all external

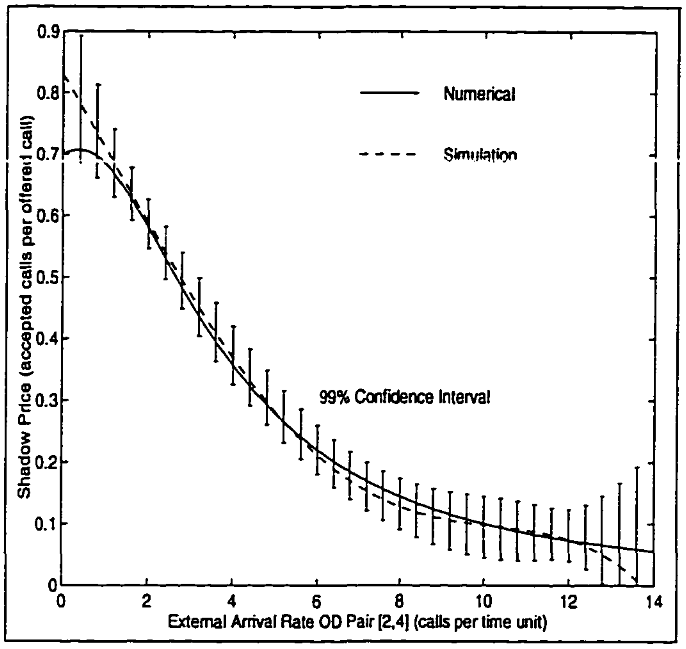

Figure 4.6: Simulated and Numerical Shadow Prices for the Five-Node Network using LLR with $T = 2$

arrival rates were kept constant except for the rate λ_{jk} of OD pair $[j,k]$. For a small constant Δ, simulations were performed using two external arrival rates for OD pair $[j,k]$, namely, $\lambda_{jk} - (\Delta/2)$ and $\lambda_{jk} + (\Delta/2)$. For these rates, the values of the rate of return, denoted W^- and W^+, respectively, were obtained from simulation. The derivative of the rate of return was then calculated from the approximation, $dW(\underline{\lambda}, \mathcal{B})/d\lambda_{jk} \approx (W^+ - W^-)/\Delta$. The value of the constant Δ used was reduced until further reduction did not significantly affect the derivative calculation. The values used in the simulation are shown in Table 3.1. The results of these simulations are presented in Figures 4.5 and 4.6 along with the results from calculations. (The confidence intervals shown are 99% confidence intervals.) It can be seen that the two are in close agreement. In Figures 3.5 and 3.7, we also show the OD pair blocking probability from this simulation as well as that obtained from calculations using the fixed point algorithm. It can be seen that the blocking probabilities match quite closely. Hereafter, we only present results of our numerical calculations.

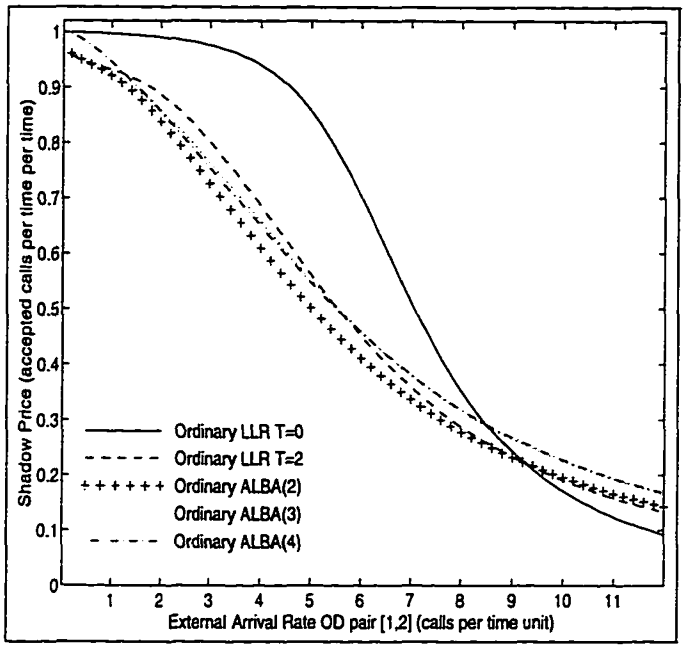

Figure 4.7: Shadow Price of Rate of Return for a Four-Node Network

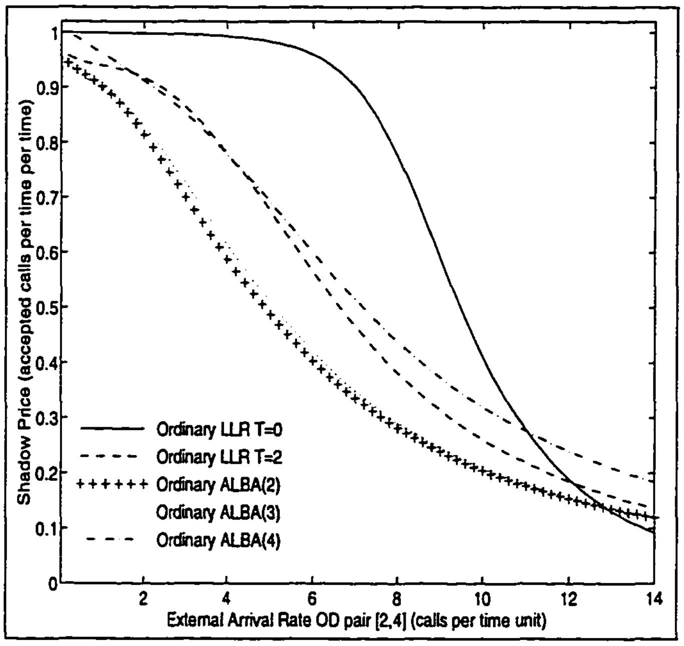

Figure 4.8: Shadow Price of Rate of Return for a Five-Node Network

As an example of the shadow price calculation, in Figures 4.7 and 4.8 the shadow prices of the network rate of return with respect to the external arrival rate of the OD pair with the smallest capacity direct link is plotted against this external arrival rate using all 5 adaptive routing schemes. Since the reward for an accepted call is 1, the shadow prices are always less than or equal to 1. Observe how the shadow price in the case of the 5-node network remains fairly constant for LLR even for arrival rates greater than 4 which is the direct link capacity in this case. This indicates that despite the overflow from the direct link of this OD pair, the existing traffic of the other OD pairs is not significantly affected. Note however that this does not hold true for LLR with trunk reservation and ALBA. This is because all these schemes have some form of trunk reservation; at the levels of traffic where the adjacent direct links reach their trunk reservation limits they stop accepting alternately routed traffic, thereby reducing the derivative of the rate of return.

4.4 Shadow Prices for RALBA

Following the same procedure as that for LLR we obtain the following expressions.

$$\frac{\partial B_{uv}(\underline{p})}{\partial p_{uv}(C_{uv})} = \frac{B_{uv}(\underline{p})}{p_{uv}(C_{uv})}, \tag{4.45}$$

$$\frac{\partial B_{uv}(\underline{p})}{\partial p_{ub}(m)} = \frac{B_{uv}(\underline{p})\,(1 - P_{bv}(K-1))}{[1 - (1 - P_{ub}(K-1))\,(1 - P_{bv}(K-1))]}, \qquad \forall m \in \mathcal{A}^{ub}_{K-1}. \tag{4.46}$$

Similarly for link (b, v). For the following expressions we use $\underline{\Phi}$ as the matrix whose values are determined by (3.9), $\underline{\Psi}$ that matrix whose values are determined by (3.9). First we indicate the dependance of the offered traffic to a link (r, s), as $\alpha_{rs}(\underline{\lambda}, \underline{p}, \underline{\Phi}, \underline{\Psi}, n)$ then we write the derivatives using the functional forms found previously. Now considering equation (4.8) but taking into account that we are talking about aggregate states, we have.

$$\frac{dp_{rs}(\underline{\alpha}_{rs}, m)}{d\lambda_{jk}} = \sum_{n=0}^{K-1} \frac{\partial p_{rs}(\underline{\alpha}_{rs}, m)}{\partial \alpha_{rs}(n)} \cdot \frac{d\alpha_{rs}(\underline{\lambda}, \underline{p}, \underline{\Phi}, \underline{\Psi}, n)}{d\lambda_{jk}}. \tag{4.47}$$

Now let $r_0^{ij} = C_{ij}$ and $r_K^{ij} = 0\ \forall (i,j) \in \mathcal{L}$ then we have from (3.10).

$$\begin{aligned} \frac{\partial p_{ij}(\underline{\alpha}_{ij}, 0)}{\partial \alpha_{ij}(n)} &= -p_{ij}(\underline{\alpha}_{ij}, 0) \Bigg\{ \mathcal{I}_{\{n=0\}} \sum_{x=0}^{C_{ij}-r_1^{ij}-1} p_{ij}(\underline{\alpha}_{ij}, x) \\ &\quad + \frac{\mathcal{I}_{\{n>0\}}}{\alpha_{ij}(\underline{\lambda}, \underline{p}, \underline{\Phi}, \underline{\Psi}, n)} \sum_{x=C_{ij}-r_n^{ij}+1}^{C_{ij}-r_{n+1}^{ij}} (x - C_{ij} + r_n^{ij}) p_{ij}(\underline{\alpha}_{ij}, x) \\ &\quad + \frac{r_n^{ij} - r_{n+1}^{ij}}{\alpha_{ij}(\underline{\lambda}, \underline{p}, \underline{\Phi}, \underline{\Psi}, n)} \sum_{u=n+1}^{K-1} \sum_{x=C_{ij}-r_u^{ij}+1}^{C_{ij}-r_{u+1}^{ij}} p_{ij}(\underline{\alpha}_{ij}, x) \Bigg\}, \end{aligned} \tag{4.48}$$

where $\mathcal{I}_{\{n=0\}} = 1$, if $n = 0$ and 0 otherwise, and $\mathcal{I}_{\{n>0\}} = 1$, if $n = 0$ and 0 otherwise. Now let $m \in \mathcal{A}_w^{ij}$, i.e., $C_{ij} - r_w^{ij} \le m < C_{ij} - r_{w+1}^{ij}$ then consider the case $w \le n$

$$\frac{\partial p_{ij}(\underline{\alpha}_{ij}, m)}{\partial \alpha_{ij}(n)} = \frac{p_{ij}(\underline{\alpha}_{ij}, m)}{p_{ij}(\underline{\alpha}_{ij}, 0)} \cdot \frac{\partial p_{ij}(\underline{\alpha}_{ij}, 0)}{\partial \alpha_{ij}(n)}. \tag{4.49}$$

Now consider the case $n \le w$, $m > 0$ to get from (3.10), (3.11) and (3.12), and define $g_{ij}(\underline{p}, \underline{\alpha}_{ij}) = \frac{p_{ij}(\underline{\alpha}_{ij}, m)}{p_{ij}(\underline{\alpha}_{ij}, 0)} \cdot \frac{\partial p_{ij}(\underline{\alpha}_{ij}, 0)}{\partial \alpha_{ij}(n)}$, then we get.

$$\frac{\partial p_{ij}(\underline{\alpha}_{ij}, m)}{\partial \alpha_{ij}(n)} = p_{ij}(\underline{\alpha}_{ij}, m-1) + g_{ij}(\underline{p}, \underline{\alpha}_{ij}), \qquad 0 < m \le C_{ij} - r_1^{ij}. \tag{4.50}$$

$$\begin{aligned} \frac{\partial p_{ij}(\underline{\alpha}_{ij}, m)}{\partial \alpha_{ij}(n)} &= \mathcal{I}_{\{n>0\}} \Bigg\{ g_{ij}(\underline{p}, \underline{\alpha}_{ij}) \\ &\quad + \frac{m - C_{ij} + r_n^{ij}}{\alpha_{ij}(\underline{\lambda}, \underline{p}, \underline{\Phi}, \underline{\Psi}, n)} p_{ij}(\underline{\alpha}_{ij}, m) \Bigg\}, \quad C_{ij} - r_n^{ij} < m \le C_{ij} - r_{n+1}^{ij}. \end{aligned} \tag{4.51}$$

$$\begin{aligned} \frac{\partial p_{ij}(\underline{\alpha}_{ij}, m)}{\partial \alpha_{ij}(n)} &= g_{ij}(\underline{p}, \underline{\alpha}_{ij}) \\ &\quad + \frac{r_n^{ij} - r_{n+1}^{ij}}{\alpha_{ij}(\underline{\lambda}, \underline{p}, \underline{\Phi}, \underline{\Psi}, n)} p_{ij}(\underline{\alpha}_{ij}, m), \qquad C_{ij} - r_{n+1}^{ij} < m. \end{aligned} \tag{4.52}$$

Now, from (3.13) and (3.17), we get the following total derivative showing the dependence of $\underline{\Phi}$ and $\underline{\Psi}$ on $\underline{p}$.

$$\begin{aligned}
\frac{d\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{d\lambda_{ij}} &= \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial\lambda_{ij}} \\
&+ \sum_{(u,v)\in S_{ab}}\sum_{x\in\Omega_{uv}} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial p_{uv}(x)}\frac{dp_{uv}(\underline{\alpha}_{uv},x)}{d\lambda_{ij}} \\
&+ \sum_{(u,v)\in S_{ab}}\sum_{\substack{d\in T_{uv}\\ d\neq b}}\sum_{G=0}^{K-1} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial\Phi^d_{uv}(G)}\frac{d\Phi^d_{uv}(\underline{p},G)}{d\lambda_{ij}} \\
&+ \sum_{(u,v)\in S_{ab}}\sum_{\substack{d\in T_{uv}\\ d\neq b}}\sum_{G=0}^{K-2} \frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial\Psi^d_{uv}(G+1)}\frac{d\Psi^d_{uv}(\underline{p},G+1)}{d\lambda_{ij}} \qquad (4.53)
\end{aligned}$$

From (3.13) and (3.17) we get

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial\lambda_{ij}} = 1 \qquad (a,b)=(i,j), \quad \forall n, \qquad (4.54)$$

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial\lambda_{ij}} = p_{ij}(C_{ij})\nu_{ij,b}(n), \qquad 0\le n < K-1, \quad (i,j)\in S_{ab}, \qquad (4.55)$$

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial\lambda_{ij}} = 1 \qquad (a,b)=(i,j), \quad n=K-1, \qquad (4.56)$$

$$\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial p_{ab}(C_{ae})} = \lambda_{ae}\nu_{ae,b}(n). \qquad (4.57)$$

Let $(b,e)\in S_{ab}$, $x\in\mathcal{A}^{be}_w$, $0\le g\le K-2$ and let $G=\max(n,g)$, then

$$\begin{aligned}
\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial p_{be}(x)} &= \lambda_{ae}p_{ae}(\underline{\alpha}_{ae},C_{ae}) \qquad (4.58)\\
&\Bigg\{\prod_{\substack{s\in T_{ae}\\ s\neq b}}\Psi^s_{ae}(\underline{p},G+1) \\
&+ \frac{1}{2}\sum_{\substack{d\in T_{ae}\\ d\neq b}}\Phi^d_{ae}(\underline{p},G)\prod_{\substack{s\in T_{ae}\\ s\neq b,d}}\Psi^s_{ae}(\underline{p},G+1) \\
&+ \frac{1}{3}\sum_{\substack{d\in T_{ae}\\ d\neq b}}\Phi^d_{ae}(\underline{p},G)\sum_{\substack{t\in T_{ae}\\ t\neq b}}\Phi^t_{ae}(\underline{p},G)\prod_{\substack{s\in T_{ae}\\ s\neq b,d,t}}\Psi^s_{ae}(\underline{p},G+1)+\cdots
\end{aligned}$$

$$\cdot \quad \cdots + \frac{1}{N-2} \prod_{\substack{s \in T_{ae} \\ s \neq b}} \Phi^s_{ae}(\underline{p}, G) \Bigg\}.$$

Let $(a,e) \in S_{ab}$, $n < H$,where H is the aggregate state of link (b,e) and let $G = \max(n, H) = H$, then from (3.13) and (3.17)

$$\begin{aligned}
\frac{\partial \alpha_{ab}(\underline{\lambda}, \underline{p}, \underline{\Phi}, \underline{\Psi}, n)}{\partial \Phi^d_{ae}(G)} &= \lambda_{ae} p_{ae}(\underline{\alpha}_{ae}, C_{ae}) P_{be}(G) \qquad (4.59) \\
& \Bigg\{ \frac{1}{2} \prod_{\substack{s \in T_{ae} \\ s \neq b,d}} \Psi^s_{ae}(\underline{p}, G+1) \\
&+ \frac{2}{3} \sum_{\substack{s \in T_{ae} \\ s \neq b,d}} \Phi^s_{ae}(\underline{p}, G) \prod_{\substack{v \in T_{ae} \\ v \neq b,d,s}} \Psi^v_{ae}(\underline{p}, G+1) \\
&+ \frac{3}{4} \sum_{\substack{v \in T_{ae} \\ v \neq b,d}} \Phi^v_{ae}(\underline{p}, G) \sum_{\substack{t \in T_{ae} \\ t \neq b,d,v}} \Phi^t_{ae}(\underline{p}, G) \prod_{\substack{s \in T_{ae} \\ s \neq b,d,t,v}} \Psi^s_{ae}(\underline{p}, G+1) + \cdots \\
&\cdot \quad \cdots + \frac{1}{N-2} \prod_{\substack{s \in T_{ae} \\ s \neq b,d}} \Phi^s_{ae}(\underline{p}, G) \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \alpha_{ab}(\underline{\lambda}, \underline{p}, \underline{\Phi}, \underline{\Psi}, n)}{\partial \Psi^d_{ae}(G+1)} &= \lambda_{ae} p_{ae}(\underline{\alpha}_{ae}, C_{ae}) P_{be}(G) \qquad (4.60) \\
& \Bigg\{ \prod_{\substack{s \in T_{ae} \\ s \neq b,d}} \Psi^s_{ae}(\underline{p}, G+1) \\
&+ \frac{1}{2} \sum_{\substack{s \in T_{ae} \\ s \neq b,d}} \Phi^s_{ae}(\underline{p}, G) \prod_{\substack{v \in T_{ae} \\ v \neq b,d,s}} \Psi^v_{ae}(\underline{p}, G+1) \\
&+ \frac{1}{3} \sum_{\substack{v \in T_{ae} \\ v \neq b,d}} \Phi^v_{ae}(\underline{p}, G) \sum_{\substack{t \in T_{ae} \\ t \neq b,d,v}} \Phi^t_{ae}(\underline{p}, G) \prod_{\substack{s \in T_{ae} \\ s \neq b,d,t,v}} \Psi^s_{ae}(\underline{p}, G+1) + \cdots \\
&\cdot \quad \cdots + \frac{1}{N-3} \sum_{\substack{v \in T_{ae} \\ v \neq b,d}} \Phi^v_{ae}(\underline{p}, G) \\
&\cdot \quad \Bigg(\sum_{\substack{t \in T_{ae} \\ t \neq b,d,v}} \Phi^t_{ae}(\underline{p}, G) \cdots \prod_{\substack{s \in T_{ae} \\ s \neq b,d}} \Psi^s_{ae}(\underline{p}, G+1) \Bigg) \Bigg\}.
\end{aligned}$$

Now with $n \geq H$, where H is the aggregate state of link (b,e) and $G = \max(n, H) = n$, then

$$\begin{aligned}\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial\Phi^d_{ae}(n)} &= \lambda_{ae}p_{ae}(\underline{\alpha}_{ae},C_{ae})\sum_{m=0}^{C_{be}-r^{be}_{n+1}-1} p_{be}(\underline{\alpha}_{be},m) \\ &\quad \left\{\frac{1}{2}\prod_{\substack{s\in\mathcal{T}_{ae}\\ s\neq b,d}}\Psi^s_{ae}(\underline{p},n+1)\right. \\ &+ \frac{2}{3}\sum_{\substack{s\in\mathcal{T}_{ae}\\ s\neq b,d}}\Phi^s_{ae}(\underline{p},n)\prod_{\substack{v\in\mathcal{T}_{ae}\\ v\neq b,d,s}}\Psi^v_{ae}(\underline{p},n+1) \\ &+ \frac{3}{4}\sum_{\substack{v\in\mathcal{T}_{ae}\\ v\neq b,d}}\Phi^v_{ae}(\underline{p},n)\sum_{\substack{t\in\mathcal{T}_{ae}\\ t\neq b,d,v}}\Phi^t_{ae}(\underline{p},n)\prod_{\substack{s\in\mathcal{T}_{ae}\\ s\neq b,d,t,v}}\Psi^s_{ae}(\underline{p},n+1)+\cdots \\ &\cdot\left.\cdots+\frac{1}{N-2}\prod_{\substack{s\in\mathcal{T}_{ae}\\ s\neq b,d}}\Phi^s_{ae}(\underline{p},n)\right\},\end{aligned} \tag{4.61}$$

$$\begin{aligned}\frac{\partial\alpha_{ab}(\underline{\lambda},\underline{p},\underline{\Phi},\underline{\Psi},n)}{\partial\Psi^d_{ae}(n+1)} &= \lambda_{ae}p_{ae}(\underline{\alpha}_{ae},C_{ae})\sum_{m=0}^{C_{be}-r^{be}_{n+1}-1} p_{be}(\underline{\alpha}_{be},m) \\ &\quad \left\{\prod_{\substack{s\in\mathcal{T}_{ae}\\ s\neq b,d}}\Psi^s_{ae}(\underline{p},n+1)\right. \\ &+ \frac{1}{2}\sum_{\substack{s\in\mathcal{T}_{ae}\\ s\neq b,d}}\Phi^s_{ae}(\underline{p},n)\prod_{\substack{v\in\mathcal{T}_{ae}\\ v\neq b,d,s}}\Psi^v_{ae}(\underline{p},n+1) \\ &+ \frac{1}{3}\sum_{\substack{v\in\mathcal{T}_{ae}\\ v\neq b,d}}\Phi^v_{ae}(\underline{p},n)\sum_{\substack{t\in\mathcal{T}_{ae}\\ t\neq b,d,v}}\Phi^t_{ae}(\underline{p},n)\prod_{\substack{s\in\mathcal{T}_{ae}\\ s\neq b,d,t,v}}\Psi^s_{ae}(\underline{p},n+1)+\cdots \\ &\cdot\cdots+\frac{1}{N-3}\sum_{\substack{v\in\mathcal{T}_{ae}\\ v\neq b,d}}\Phi^v_{ae}(\underline{p},n) \\ &\cdot\left.\left(\sum_{\substack{t\in\mathcal{T}_{ae}\\ t\neq b,d,v}}\Phi^t_{ae}(\underline{p},n)\cdots\prod_{\substack{s\in\mathcal{T}_{ae}\\ s\neq b,d}}\Psi^s_{ae}(\underline{p},n+1)\right)\right\}.\end{aligned} \tag{4.62}$$

From (3.9) we have

$$\frac{d\Phi^d_{uv}(\underline{p},G)}{d\lambda_{ij}}=\sum_{(a,b)\in S_{uv}}\sum_{m\in\Omega_{ab}}\frac{\partial\Phi^d_{uv}(\underline{p},G)}{\partial p_{ab}(m)}\frac{dp_{ab}(\underline{\alpha}_{ab},m)}{d\lambda_{ij}}, \tag{4.63}$$

where

$$\frac{\partial\Phi^d_{uv}(\underline{p},G)}{\partial p_{ud}(m)} = \mathcal{I}_{\{m\le C_{ud}-r^{ud}_G-1\}}\sum_{x=0}^{C_{dv}-r^{dv}_{G+1}-1} p_{dv}(\underline{\alpha}_{dv},x) \tag{4.64}$$

$$
\begin{aligned}
& - \mathcal{I}_{\{m \leq C_{ud} - r_G^{ud} - 1\}} \sum_{x=0}^{C_{dv} - r_G^{dv} - 1} p_{dv}(\underline{\alpha}_{dv}, x) \\
& + \mathcal{I}_{\{C_{ud} - r_G^{ud} \leq m \leq C_{ud} - r_{G+1}^{ud} - 1\}} \sum_{x=0}^{C_{dv} - r_{G+1}^{dv} - 1} p_{dv}(\underline{\alpha}_{dv}, x),
\end{aligned}
$$

and from (3.9)

$$
\frac{d\Psi_{uv}^{d}(\underline{p}, G+1)}{d\lambda_{ij}} = \sum_{(a,b)\in S_{uv}} \sum_{m \in \Omega_{ab}} \frac{\partial \Psi_{uv}^{d}(\underline{p}, G+1)}{\partial p_{ab}(m)} \frac{dp_{ab}(\underline{\alpha}_{ab}, m)}{d\lambda_{ij}}, \tag{4.65}
$$

where

$$
\frac{\partial \Psi_{uv}^{d}(\underline{p}, G+1)}{\partial p_{ud}(m)} = -\mathcal{I}_{\{m \leq C_{ud} - r_{G+1}^{ud} - 1\}} \sum_{x=0}^{C_{dv} - r_{G+1}^{dv} - 1} p_{dv}(\underline{\alpha}_{dv}, x). \tag{4.66}
$$

Substituting equations (4.54) - (4.66) into (4.53) and the result together with equations (4.49) - (4.53) into equation (4.47), we get the set of simultaneous linear equations, as in the case of ALBA and LLR, to be solved to find the shadow price. In Figure 4.9, the shadow price of the rate of return with respect to the external arrival rate of OD pair $[1,2]$ for the 4-node network is presented for all the routing schemes analyzed. It can be seen that the value of the shadow price decreases more rapidly when using RALBA than any of the other routing schemes. Other results for RALBA can be found in Chapter 5 where it is compared to LLR using sum capacity.

4.5 Shadow Prices for MLLR

The shadow prices considered are the derivatives of the rate of return with respect to load sharing coefficients, i.e., β_{ij} $\forall [i,j] \in \mathcal{O}$, when the exogenous traffics are kept constant, where the rate of return determines the revenue generated from the carried traffic in the network based on exogenous arrival rates and OD pair blocking

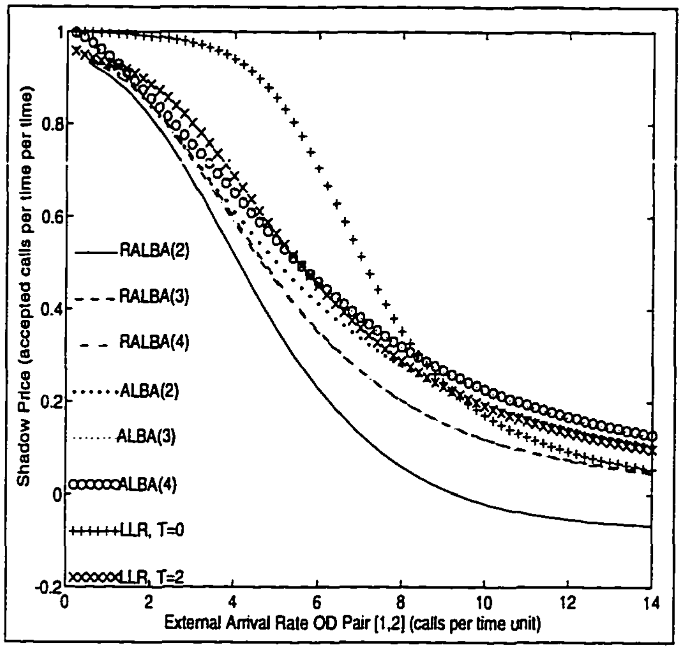

Figure 4.9: Shadow Price of Rate of Return for a Four-Node Network

probabilities. In this section, we give the expressions for the shadow prices for the blocking probabilities given by the fixed point equations (3.18), (3.19) and (3.20) for the evaluation of MLLR.

In the case of Fixed Alternate Routing (FAR), the shadow price calculation of [35] requires the blocking probability of every link in the network. By contrast, in the case of adaptive routing schemes, the calculation of the shadow price of the blocking probability for an OD pair depends on the probability of all the states of all the links in the network. This makes the calculation more complicated, especially in the case of asymmetric networks and we provide some results for small networks with four and five nodes.

Assume that for each accepted call on OD pair $[j, k]$, the expected revenue is w_{jk}. We first write the expression for the rate of return, (when blocking probability is the performance measure, all the w_{jk}'s are 1), let $\underline{p}$ be a vector which is obtained by the concatenation of the stationary probability vectors of each link, let $\underline{\alpha}_{ij}$ be a vector containing the offered traffic for all the states of link (i,j) and let $\underline{\mathcal{B}}$ be the vector of OD pair blocking probabilities. Then we have

$$W(\underline{\mathcal{B}}) = \sum_{[j,k]\in\mathcal{O}} \lambda_{jk} w_{jk} \left(1 - B_{jk}(\underline{p})\right), \tag{4.67}$$

$$\frac{dW(\underline{\mathcal{B}})}{d\beta_{jk}(u)} = \sum_{[r,s]\in\mathcal{O}} \left[\frac{\partial W(\underline{\mathcal{B}})}{\partial B_{rs}} \frac{dB_{rs}(\underline{p})}{d\beta_{jk}(u)}\right], \tag{4.68}$$

$$\frac{\partial W(\underline{\mathcal{B}})}{\partial B_{rs}} = -\lambda_{rs} w_{rs}, \tag{4.69}$$

$$\frac{dB_{rs}(\underline{p})}{d\beta_{jk}(u)} = \sum_{(a,b)\in\mathcal{L}} \frac{\partial B_{rs}(\underline{p})}{\partial p_{ab}(0)} \cdot \frac{dp_{ab}(\underline{\alpha}_{ab}, 0)}{d\beta_{jk}(u)}, \tag{4.70}$$

where $W(\underline{\mathcal{B}})$ is the rate of return from the network and is written to show its dependence on $\underline{\mathcal{B}}$, $p_{ab}(\underline{\alpha}_{ab}, 0)$ is defined in (3.18) and is written to explicitly show its dependence on $\underline{\alpha}_{ab}$, and $B_{rs}(\underline{p})$ is defined in (3.24) and is written to explicitly show its dependence on $\underline{p}$. Note that since the equilibrium probability vectors and blocking probabilities we use are those given by the fixed point models of Section 2, the W in (4.67) is an *approximate* rate of return. As a result, the shadow prices we calculate are also approximations to the actual shadow prices.

The fixed point model describes the $\underline{p}$, as an implicit function of the load sharing coefficients. $\underline{\mathcal{B}}$ is, in turn, a function of $\underline{p}$ and thereby an implicit function of the load sharing coefficients. Consequently, $W(\underline{\mathcal{B}})$ is also an implicit function of these coefficients. We therefore undertake a careful and extensive effort to obtain relations of total and partial derivatives of OD pair blocking probabilities by differentiating the fixed point equations. These relations are manipulated to obtain a system of linear equations in the derivatives of the OD pair blocking probabilities with respect to the load sharing coefficients. It is known that the blocking probability solutions to the fixed point equations for alternate routing may not be unique [35]. Even in this case, under a maximum rank condition on the Jacobian of the fixed point equations, the implicit function theorem [15], confirms the existence of a locally differentiable function whose value at the point in question is the blocking probability. In all that follows, we assume that this condition is met and therefore all the differentiation operations we undertake are legitimate.

Using (3.24) as the functional form for the blocking probability for an OD pair, we can see that it depends on the probabilities of those links which form the paths for the routes available for the OD pair in question when those links are fully occupied. Hence we obtain

$$\frac{\partial B_{rs}(\underline{p})}{\partial p_{rs}(0)} = \frac{B_{rs}(\underline{p})}{p_{rs}(\underline{\alpha}_{rs}, 0)}, \tag{4.71}$$

$$\frac{\partial B_{rs}(\underline{p})}{\partial p_{rd}(0)} = \frac{B_{rs}(\underline{p})\Big[1 - p_{ds}(\underline{\alpha}_{ds}, 0)\Big]}{1 - \Big[1 - p_{rd}(\underline{\alpha}_{rd}, 0)\Big]\Big[1 - p_{ds}(\underline{\alpha}_{ds}, 0)\Big]}. \tag{4.72}$$

similar derivatives can be obtained for $\frac{\partial B_{rs}(\underline{p})}{\partial p_{ds}(0)}$. The second term in (4.70) can be computed as

$$\frac{dp_{ab}(\underline{\alpha}_{ab}, 0)}{d\beta_{jk}(u)} = \sum_{n=1}^{C_{ab}} \frac{\partial p_{ab}(\underline{\alpha}_{ab}, 0)}{\partial \alpha_{ab}(n)} \cdot \frac{d\alpha_{ab}(\underline{\beta}, \underline{P}, n)}{d\beta_{jk}(u)}, \tag{4.73}$$

where $\alpha_{ab}(\underline{\beta}, \underline{P}, n)$ is defined in (3.20) and is written to show explicitly its dependence on $\underline{\beta}$ and $\underline{P}$, where $\underline{\beta}$ is the concatenation of the load sharing coefficients allocated to each route per OD pair and $\underline{P}$ is the concatenation of the probabilities given by equations (3.21) to (3.23) for each route and each OD pair.

Now, we get each term of (4.73) for every n. The evaluation of $\frac{\partial p_{ab}(\underline{\alpha}_{ab},m)}{\partial \alpha_{ab}(n)}$ for $C_{ab} \geq n > m \geq 0$, is given by

$$\frac{\partial p_{ab}(\underline{\alpha}_{ab}, m)}{\partial \alpha_{ab}(n)} = \frac{p_{ab}(\underline{\alpha}_{ab}, m)}{\alpha_{ab}(\underline{\beta}, \underline{P}, n)} \sum_{j=n}^{C_{ab}} p_{ab}(\underline{\alpha}_{ab}, j). \tag{4.74}$$

and for $C_{ab} \geq m \geq n \geq 1$ we get

$$\frac{\partial p_{ab}(\underline{\alpha}_{ab}, m)}{\partial \alpha_{ab}(n)} = -\frac{p_{ab}(\underline{\alpha}_{ab}, m)}{\alpha_{ab}(\underline{\beta}, \underline{P}, n)} \Big[1 - \sum_{j=n}^{C_{ab}} p_{ab}(\underline{\alpha}_{ab}, j)\Big]. \tag{4.75}$$

To complete the evaluation of the right hand side of (4.73) we compute the derivative of $\alpha_{ab}(\underline{\beta}, \underline{P}, n)$ with respect to $\beta_{jk}(u)$. From (3.20), and (3.21) through (3.23), we can write.

$$\begin{aligned}
\frac{d\alpha_{ab}(\underline{\beta}, \underline{P}, n)}{d\beta_{jk}(u)} &= \frac{\partial \alpha_{ab}(\underline{\beta}, \underline{P}, n)}{\partial \beta_{jk}(u)} \\
&+ \sum_{\substack{w:[a,w]\in\mathcal{O} \\ w\neq b}} \frac{\partial \alpha_{ab}(\underline{\beta}, \underline{P}, n)}{\partial p_{aw}(0)} \frac{dp_{aw}(\underline{\alpha}_{aw}, 0)}{d\beta_{jk}(u)} \\
&+ \sum_{\substack{w:[w,b]\in\mathcal{O} \\ w\neq a}} \frac{\partial \alpha_{ab}(\underline{\beta}, \underline{P}, n)}{\partial p_{wb}(0)} \frac{dp_{wb}(\underline{\alpha}_{wb}, 0)}{d\beta_{jk}(u)} \\
&+ \sum_{w:[a,w]\in\mathcal{O}} \sum_{\substack{d\in S_{aw} \\ d\neq b}} \frac{\partial \alpha_{ab}(\underline{\beta}, \underline{P}, n)}{\partial P_{aw}(d,b,n)} \frac{dP_{aw}(d,b,n)}{d\beta_{jk}(u)} \\
&+ \sum_{\substack{w:[w,b]\in\mathcal{O} \\ w\neq a}} \sum_{\substack{d\in S_{wb} \\ d\neq a}} \frac{\partial \alpha_{ab}(\underline{\beta}, \underline{P}, n)}{\partial P_{wb}(d,a,n)} \frac{dP_{wb}(d,a,n)}{d\beta_{jk}(u)}.
\end{aligned} \tag{4.76}$$

From (3.20) we can obtain for $n > 0$ and $w \in \mathcal{T}_{ab}$ the following

$$\frac{\partial \alpha_{ab}(\underline{\beta}, \underline{P}, n)}{\partial p_{aw}(0)} = -\lambda_{wb}\beta_{wb}(a), \tag{4.77}$$

for $w \neq b$, $w \in \mathcal{T}_{ab}$ and $d \neq b$, $d \in \mathcal{T}_{aw}$

$$\frac{\partial \alpha_{ab}(\underline{\beta}, \underline{P}, n)}{\partial P_{aw}(d,b,n)} = \lambda_{aw}\beta_{aw}(d), \tag{4.78}$$

similarly for $\frac{\partial \alpha_{ab}(\underline{\beta}, \underline{P}, n)}{\partial p_{wb}(0)}$ and $\frac{\partial \alpha_{ab}(\underline{\beta}, \underline{P}, n)}{\partial P_{wb}(d,a,n)}$. Now, from (3.20) we can obtain the derivative of $\alpha_{ab}(\underline{\beta}, \underline{P}, n)$ with respect to the load sharing coefficients as follows

$$\frac{\partial \alpha_{ab}(\underline{\beta}, \underline{P}, n)}{\partial \beta_{jk}(u)} = \begin{cases} \lambda_{ab}, & \beta_{jk}(u) = \beta_{ab}(b), \\ \lambda_{aw}[1 - p_{wb}(\underline{\alpha}_{wb}, 0)], & \beta_{jk}(u) = \beta_{aw}(b),\ w \neq b, \\ \lambda_{wb}[1 - p_{aw}(\underline{\alpha}_{aw}, 0)], & \beta_{jk}(u) = \beta_{wb}(a),\ w \neq a, \\ \lambda_{aw} P_{aw}(u, b, n), & \beta_{jk}(u) = \beta_{aw}(u),\ u \neq b, \\ \lambda_{wb} P_{wb}(u, a, n), & \beta_{jk}(u) = \beta_{wb}(u),\ u \neq a,\ w \neq a. \end{cases} \tag{4.79}$$

To complete the evaluation of (4.76), consider first that link (a, b) is fixed and it is in state n, then we need to obtain the following derivatives

$$\begin{aligned} \frac{dP_{aw}(d,b,n)}{d\beta_{jk}(u)} &= \sum_{v:(a,v)\in\mathcal{L}} \sum_{x=0}^{C_{av}} \frac{\partial P_{aw}(d,b,n)}{\partial p_{av}(x)} \frac{dp_{av}(\underline{\alpha}_{av}, x)}{d\beta_{jk}(u)} \\ &+ \sum_{v:(v,b)\in\mathcal{L}} \sum_{x=0}^{C_{vb}} \frac{\partial P_{aw}(d,b,n)}{\partial p_{vb}(x)} \frac{dp_{vb}(\underline{\alpha}_{vb}, x)}{d\beta_{jk}(u)} \\ &+ \sum_{v:(v,w)\in\mathcal{L}} \sum_{x=0}^{C_{vw}} \frac{\partial P_{aw}(d,b,n)}{\partial p_{vw}(x)} \frac{dp_{vw}(\underline{\alpha}_{vw}, x)}{d\beta_{jk}(u)}, \end{aligned} \tag{4.80}$$

similarly for $P_{wb}(d, a, n)$. In the following, we obtain the partial derivatives needed in (4.80) depending on the different cases arising from each of the equations (3.21), (3.22) and (3.23). Assume the conditions for equation (3.21), i.e., OD pair $[a, w]$ has as first-choice route the path (a, d), (d, w), $d \neq b, w$, when link (a, b) is in state n fixed, then

$$\frac{\partial P_{aw}(d,b,n)}{\partial p_{ab}(n)} = 0, \qquad \forall\ n. \tag{4.81}$$

Now, obtain $\frac{\partial P_{aw}(d,b,n)}{\partial p_{av}(x)}$, $\forall\ v$, such that the link $(a, v) \in \mathcal{L}$, $v \neq b$ and for the states $x = 0, 1, \ldots, C_{av}$ as follows

$$\frac{\partial P_{aw}(d,b,n)}{\partial p_{ad}(0)} = \frac{[1 - p_{dw}(\underline{\alpha}_{dw}, 0)] P_{aw}(d,b,n)}{1 - [1 - p_{ad}(\underline{\alpha}_{ad}, 0)][1 - p_{dw}(\underline{\alpha}_{dw}, 0)]}, \tag{4.82}$$

$$\frac{\partial P_{aw}(d,b,n)}{\partial p_{dw}(0)} = \frac{[1-p_{ad}(\underline{\alpha}_{ad},0)]P_{aw}(d,b,n)}{1-[1-p_{ad}(\underline{\alpha}_{ad},0)][1-p_{dw}(\underline{\alpha}_{dw},0)]}. \tag{4.83}$$

Consider that $n \geq x > 0$ then we have

$$\begin{aligned} \frac{\partial P_{aw}(d,b,n)}{\partial p_{wb}(x)} &= \Big\{1-\Big[1-p_{ad}(\underline{\alpha}_{ad},0)\Big]\Big[1-p_{dw}(\underline{\alpha}_{dw},0)\Big]\Big\} \\ &\cdot \left(1-\sum_{z=x}^{C_{aw}}\mathcal{I}_{\{w\in A^-_{aw}(ab)\}}p_{aw}(\underline{\alpha}_{aw},z)\right) \\ &\cdot \left(1-\sum_{z=x+1}^{C_{aw}}\mathcal{I}_{\{w\in A^+_{aw}(ab)\}}p_{aw}(\underline{\alpha}_{aw},z)\right) \\ &\cdot \prod_{\substack{u\in A^-_{aw}(ab)\\ u\neq w,d}}\left(1-\left[\sum_{s=x}^{C_{au}}p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{r=x}^{C_{uw}}p_{uw}(\underline{\alpha}_{uw},r)\right]\right) \\ &\cdot \prod_{\substack{u\in A^+_{aw}(ab)\\ u\neq w,d}}\left(1-\left[\sum_{s=x+1}^{C_{au}}p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{r=x+1}^{C_{uw}}p_{uw}(\underline{\alpha}_{uw},r)\right]\right). \end{aligned} \tag{4.84}$$

Now for the case $x > n > 0$ we have

$$\begin{aligned} \frac{\partial P_{aw}(d,b,n)}{\partial p_{wb}(x)} &= \Big\{1-\Big[1-p_{ad}(\underline{\alpha}_{ad},0)\Big]\Big[1-p_{dw}(\underline{\alpha}_{dw},0)\Big]\Big\} \\ &\cdot \left(1-\sum_{z=n}^{C_{aw}}\mathcal{I}_{\{w\in A^-_{aw}(ab)\}}p_{aw}(\underline{\alpha}_{aw},z)\right) \\ &\cdot \left(1-\sum_{z=n+1}^{C_{aw}}\mathcal{I}_{\{w\in A^+_{aw}(ab)\}}p_{aw}(\underline{\alpha}_{aw},z)\right) \\ &\cdot \prod_{\substack{u\in A^-_{aw}(ab)\\ u\neq w,d}}\left(1-\left[\sum_{s=n}^{C_{au}}p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{r=n}^{C_{uw}}p_{uw}(\underline{\alpha}_{uw},r)\right]\right) \\ &\cdot \prod_{\substack{u\in A^+_{aw}(ab)\\ u\neq w,d}}\left(1-\left[\sum_{s=n+1}^{C_{au}}p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{r=n+1}^{C_{uw}}p_{uw}(\underline{\alpha}_{uw},r)\right]\right). \end{aligned} \tag{4.85}$$

For the following partial derivative consider the single link of OD pair $[a,w]$ and the case when $n > x > 0$ with x as the state of link (a,w), then we get

$$\begin{aligned} \frac{\partial P_{aw}(d,b,n)}{\partial p_{aw}(x)} = &- \Big\{1-\Big[1-p_{ad}(\underline{\alpha}_{ad},0)\Big]\Big[1-p_{dw}(\underline{\alpha}_{dw},0)\Big]\Big\} \\ &\Bigg\{ \sum_{r=1}^{x}\Big[p_{wb}(\underline{\alpha}_{wb},r)\mathcal{I}_{\{w\in A^-_{aw}(ab)\}} \end{aligned} \tag{4.86}$$

$$
\begin{aligned}
&\cdot \prod_{\substack{u\in A^-_{aw}(ab)\\ u\neq w,d}} \left(1-\left[\sum_{s=r}^{C_{au}} p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{z=r}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\right]\right)\\
&\cdot \prod_{\substack{u\in A^+_{aw}(ab)\\ u\neq w,d}} \left(1-\left[\sum_{s=r+1}^{C_{au}} p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{z=r+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\right]\right)\Bigg]\\
&+\sum_{r=1}^{x-1}\Bigg[p_{wb}(\underline{\alpha}_{wb},r)\mathcal{I}_{\{w\in A^+_{aw}(ab)\}}\\
&\cdot \prod_{\substack{u\in A^-_{aw}(ab)\\ u\neq w,d}} \left(1-\left[\sum_{s=r}^{C_{au}} p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{z=r}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\right]\right)\\
&\cdot \prod_{\substack{u\in A^+_{aw}(ab)\\ u\neq w,d}} \left(1-\left[\sum_{s=r+1}^{C_{au}} p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{z=r+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\right]\right)\Bigg]\Bigg\}.
\end{aligned}
$$

and for the case $x \geq n > 0$ we have

$$
\begin{aligned}
\frac{\partial P_{aw}(d,b,n)}{\partial p_{aw}(x)} = &-\left\{1-\left[1-p_{ad}(\underline{\alpha}_{ad},0)\right]\left[1-p_{dw}(\underline{\alpha}_{dw},0)\right]\right\} \qquad (4.87)\\
&\Bigg\{\sum_{r=1}^{n}\Bigg[p_{wb}(\underline{\alpha}_{wb},r)\mathcal{I}_{\{w\in A^-_{aw}(ab)\}}\\
&\cdot \prod_{\substack{u\in A^-_{aw}(ab)\\ u\neq w,d}} \left(1-\left[\sum_{s=r}^{C_{au}} p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{z=r}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\right]\right)\\
&\cdot \prod_{\substack{u\in A^+_{aw}(ab)\\ u\neq w,d}} \left(1-\left[\sum_{s=r+1}^{C_{au}} p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{z=r+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\right]\right)\Bigg]\\
&+\sum_{r=n+1}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb},r)\mathcal{I}_{\{w\in A^-_{aw}(ab),\ x\geq n\}}\\
&\cdot \prod_{\substack{u\in A^-_{aw}(ab)\\ u\neq w,d}} \left(1-\left[\sum_{s=n}^{C_{au}} p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{z=n}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\right]\right)\\
&\cdot \prod_{\substack{u\in A^+_{aw}(ab)\\ u\neq w,d}} \left(1-\left[\sum_{s=n+1}^{C_{au}} p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{z=n+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\right]\right)\\
&+\sum_{r=1}^{n-1}\Bigg[p_{wb}(\underline{\alpha}_{wb},r)\mathcal{I}_{\{w\in A^+_{aw}(ab)\}}\\
&\cdot \prod_{\substack{u\in A^-_{aw}(ab)\\ u\neq w,d}} \left(1-\left[\sum_{s=r}^{C_{au}} p_{au}(\underline{\alpha}_{au},s)\right]\left[\sum_{z=r}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\right]\right)
\end{aligned}
$$

$$
\cdot \prod_{\substack{u \in A^+_{aw}(ab) \\ u \neq w,d}} \left(1 - \left[\sum_{s=r+1}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right] \left[\sum_{z=r+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, z)\right]\right)\Bigg]
$$

$$
+ \sum_{r=n}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb}, r) \mathcal{I}_{\{w \in A^+_{aw}(ab),\ x > n\}}
$$

$$
\cdot \prod_{\substack{u \in A^-_{aw}(ab) \\ u \neq w,d}} \left(1 - \left[\sum_{s=n}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right] \left[\sum_{z=n}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, z)\right]\right)
$$

$$
\cdot \prod_{\substack{u \in A^+_{aw}(ab) \\ u \neq w,d}} \left(1 - \left[\sum_{s=n+1}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right] \left[\sum_{z=n+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, z)\right]\right)\Bigg\}.
$$

The following are the partial derivatives of the same probability $P_{aw}(d, b, n)$, but with respect to the probability distributions for the states of those links that form a two-link alternate route for OD pair $[a, w]$, i.e., links of the form $(a, u) \in \mathcal{L}$, $u \in \mathcal{T}_{aw}$, $u \neq b, d, w$ and links of the form $(u, w) \in \mathcal{L}$ and $u \neq b, d, a$. First consider that link (a, u) is in state $x = 0, 1, \ldots, C_{au}$ and with link (a, b) in state n fixed, then we have for $n > x > 0$

$$
\frac{\partial P_{aw}(d,b,n)}{\partial p_{au}(x)} = -\Big\{1 - \big[1 - p_{ad}(\underline{\alpha}_{ad}, 0)\big]\big[1 - p_{dw}(\underline{\alpha}_{dw}, 0)\big]\Big\} \tag{4.88}
$$

$$
\Bigg\{ \mathcal{I}_{\{u \in A^-_{aw}(ab)\}} \sum_{r=1}^{x} \Bigg[p_{wb}(\underline{\alpha}_{wb}, r)
$$

$$
\cdot \left(1 - \sum_{s=r}^{C_{aw}} \mathcal{I}_{\{w \in A^-_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right)
$$

$$
\cdot \left(1 - \sum_{s=r+1}^{C_{aw}} \mathcal{I}_{\{w \in A^+_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \sum_{z=r}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, z)
$$

$$
\cdot \prod_{\substack{y \in A^-_{aw}(ab) \\ y \neq w,d,u}} \left(1 - \left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right)
$$

$$
\cdot \prod_{\substack{y \in A^+_{aw}(ab) \\ y \neq w,d}} \left(1 - \left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right)\Bigg]
$$

$$
+ \mathcal{I}_{\{u \in A^+_{aw}(ab)\}} \sum_{r=1}^{x-1} \Bigg[p_{wb}(\underline{\alpha}_{wb}, r)
$$

$$
\begin{aligned}
&\cdot \left(1-\sum_{s=r}^{C_{aw}} \mathcal{I}_{\{w\in A^-_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw},s)\right)\\
&\cdot \left(1-\sum_{s=r+1}^{C_{aw}} \mathcal{I}_{\{w\in A^+_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw},s)\right) \sum_{z=r+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\\
&\cdot \prod_{\substack{y\in A^-_{aw}(ab)\\ y\neq w,d}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\\
&\cdot \prod_{\substack{y\in A^+_{aw}(ab)\\ y\neq w,d,u}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]\Bigg\}.
\end{aligned}
$$

Now for the same case $n > x > 0$, but for the partial derivative with respect to link (u, w) in state x, we have

$$
\begin{aligned}
\frac{\partial P_{aw}(d,b,n)}{\partial p_{uw}(x)} = & -\Big\{1-\big[1-p_{ad}(\underline{\alpha}_{ad},0)\big]\big[1-p_{dw}(\underline{\alpha}_{dw},0)\big]\Big\} \qquad (4.89)\\
& \Bigg\{ \mathcal{I}_{\{u\in A^-_{aw}(ab)\}} \sum_{r=1}^{x} \Bigg[p_{wb}(\underline{\alpha}_{wb},r)\\
&\cdot \left(1-\sum_{s=r}^{C_{aw}} \mathcal{I}_{\{w\in A^-_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw},s)\right)\\
&\cdot \left(1-\sum_{s=r+1}^{C_{aw}} \mathcal{I}_{\{w\in A^+_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw},s)\right) \sum_{z=r}^{C_{au}} p_{au}(\underline{\alpha}_{au},z)\\
&\cdot \prod_{\substack{y\in A^-_{aw}(ab)\\ y\neq w,d,u}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\\
&\cdot \prod_{\substack{y\in A^+_{aw}(ab)\\ y\neq w,d}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]\\
&+ \mathcal{I}_{\{u\in A^+_{aw}(ab)\}} \sum_{r=1}^{x-1} \Bigg[p_{wb}(\underline{\alpha}_{wb},r)\\
&\cdot \left(1-\sum_{s=r}^{C_{aw}} \mathcal{I}_{\{w\in A^-_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw},s)\right)\\
&\cdot \left(1-\sum_{s=r+1}^{C_{aw}} \mathcal{I}_{\{w\in A^+_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw},s)\right) \sum_{z=r+1}^{C_{au}} p_{au}(\underline{\alpha}_{au},z)\\
&\cdot \prod_{\substack{y\in A^-_{aw}(ab)\\ y\neq w,d}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)
\end{aligned}
$$

$$\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w,d,u}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]\Bigg\}.$$

Next, the same partial derivative with respect to the same two links, (a,u) and (u,w), but the state x satisfies the condition $x \geq n > 0$, then we have

$$\begin{aligned}
\frac{\partial P_{aw}(d,b,n)}{\partial p_{au}(x)} &= -\Big\{1-\big[1-p_{ad}(\underline{\alpha}_{ad},0)\big]\big[1-p_{dw}(\underline{\alpha}_{dw},0)\big]\Big\} \qquad (4.90)\\
&\Bigg\{ \mathcal{I}_{\{u\in A^{-}_{aw}(ab)\}}\sum_{r=1}^{n}\Bigg[p_{wb}(\underline{\alpha}_{wb},r)\\
&\cdot \left(1-\sum_{s=r}^{C_{aw}}\mathcal{I}_{\{w\in A^{-}_{aw}(ab)\}}p_{aw}(\underline{\alpha}_{aw},s)\right)\\
&\cdot \left(1-\sum_{\substack{s=r+1\\ w\in A^{+}_{aw}(ab)}}^{C_{aw}}\mathcal{I}_{\{w\in A^{+}_{aw}(ab)\}}p_{aw}(\underline{\alpha}_{aw},s)\right)\sum_{z=r}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\\
&\cdot \prod_{\substack{y\in A^{-}_{aw}(ab)\\ y\neq w,d,u}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\\
&\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w,d}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]\\
&+ \mathcal{I}_{\{u\in A^{-}_{aw}(ab)\}}\sum_{\substack{r=n+1\\ u\in A^{-}_{aw}(ab)}}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb},r)\\
&\cdot \left(1-\sum_{s=n}^{C_{aw}}\mathcal{I}_{\{w\in A^{-}_{aw}(ab)\}}p_{aw}(\underline{\alpha}_{aw},s)\right)\\
&\cdot \left(1-\sum_{s=n+1}^{C_{aw}}\mathcal{I}_{\{w\in A^{+}_{aw}(ab)\}}p_{aw}(\underline{\alpha}_{aw},s)\right)\sum_{z=n}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)\\
&\cdot \prod_{\substack{y\in A^{-}_{aw}(ab)\\ y\neq w,d,u}} \left(1-\left[\sum_{s=n}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=n}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\\
&\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w,d}} \left(1-\left[\sum_{s=n+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=n+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\\
&+ \mathcal{I}_{\{u\in A^{+}_{aw}(ab)\}}\sum_{r=1}^{n-1}\Bigg[p_{wb}(\underline{\alpha}_{wb},r)\\
&\cdot \left(1-\sum_{s=r}^{C_{aw}}\mathcal{I}_{\{w\in A^{-}_{aw}(ab)\}}p_{aw}(\underline{\alpha}_{aw},s)\right)
\end{aligned}$$

$$
\begin{aligned}
&\cdot \left(1 - \sum_{s=r+1}^{C_{aw}} \mathcal{I}_{\{w \in A^{+}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \sum_{z=r+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, z) \\
&\cdot \prod_{\substack{y \in A^{-}_{aw}(ab) \\ y \neq w,d}} \left(1 - \left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \\
&\cdot \prod_{\substack{y \in A^{+}_{aw}(ab) \\ y \neq w,d,u}} \left(1 - \left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \Bigg] \\
&+ \mathcal{I}_{\{u \in A^{+}_{aw}(ab),\ x > n\}} \sum_{r=n}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb}, r) \\
&\cdot \left(1 - \sum_{s=n}^{C_{aw}} \mathcal{I}_{\{w \in A^{-}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \\
&\cdot \left(1 - \sum_{s=n+1}^{C_{aw}} \mathcal{I}_{\{w \in A^{+}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \sum_{z=n+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, z) \\
&\cdot \prod_{\substack{y \in A^{-}_{aw}(ab) \\ y \neq w,d}} \left(1 - \left[\sum_{s=n}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=n}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \\
&\cdot \prod_{\substack{y \in A^{+}_{aw}(ab) \\ y \neq w,d,u}} \left(1 - \left[\sum_{s=n+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=n+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial P_{aw}(d,b,n)}{\partial p_{uw}(x)} = & -\Big\{1 - \big[1 - p_{ad}(\underline{\alpha}_{ad}, 0)\big]\big[1 - p_{dw}(\underline{\alpha}_{dw}, 0)\big]\Big\} \qquad (4.91) \\
&\Bigg\{ \mathcal{I}_{\{u \in A^{-}_{aw}(ab)\}} \sum_{r=1}^{n} \Bigg[p_{wb}(\underline{\alpha}_{wb}, r) \\
&\cdot \left(1 - \sum_{s=r}^{C_{aw}} \mathcal{I}_{\{w \in A^{-}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \\
&\cdot \left(1 - \sum_{s=r+1}^{C_{aw}} \mathcal{I}_{\{w \in A^{+}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \sum_{z=r}^{C_{au}} p_{au}(\underline{\alpha}_{au}, z) \\
&\cdot \prod_{\substack{y \in A^{-}_{aw}(ab) \\ y \neq w,d,u}} \left(1 - \left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \\
&\cdot \prod_{\substack{y \in A^{+}_{aw}(ab) \\ y \neq w,d}} \left(1 - \left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \Bigg] \\
&+ \mathcal{I}_{\{u \in A^{-}_{aw}(ab)\}} \sum_{r=n+1}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb}, r)
\end{aligned}
$$

$$
\begin{aligned}
&\cdot \left(1 - \sum_{s=n}^{C_{aw}} \mathcal{I}_{\{w \in A^{-}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \\
&\cdot \left(1 - \sum_{s=n+1}^{C_{aw}} \mathcal{I}_{\{w \in A^{+}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \sum_{z=n}^{C_{au}} p_{au}(\underline{\alpha}_{au}, z) \\
&\cdot \prod_{\substack{y \in A^{-}_{aw}(ab) \\ y \neq w,d,u}} \left(1 - \left[\sum_{s=n}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=n}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \\
&\cdot \prod_{\substack{y \in A^{+}_{aw}(ab) \\ y \neq w,d}} \left(1 - \left[\sum_{s=n+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=n+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \\
&+ \mathcal{I}_{\{u \in A^{+}_{aw}(ab)\}} \sum_{r=1}^{n-1} \left[p_{wb}(\underline{\alpha}_{wb}, r) \right. \\
&\cdot \left(1 - \sum_{s=r}^{C_{aw}} \mathcal{I}_{\{w \in A^{-}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \\
&\cdot \left(1 - \sum_{s=r+1}^{C_{aw}} \mathcal{I}_{\{w \in A^{+}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \sum_{z=r+1}^{C_{au}} p_{au}(\underline{\alpha}_{au}, z) \\
&\cdot \prod_{\substack{y \in A^{-}_{aw}(ab) \\ y \neq w,d}} \left(1 - \left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \\
&\cdot \left. \prod_{\substack{y \in A^{+}_{aw}(ab) \\ y \neq w,d,u}} \left(1 - \left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \right] \\
&+ \mathcal{I}_{\{u \in A^{+}_{aw}(ab),\ x > n\}} \sum_{r=n}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb}, r) \\
&\cdot \left(1 - \sum_{s=n}^{C_{aw}} \mathcal{I}_{\{w \in A^{-}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \\
&\cdot \left(1 - \sum_{s=n+1}^{C_{aw}} \mathcal{I}_{\{w \in A^{+}_{aw}(ab)\}} p_{aw}(\underline{\alpha}_{aw}, s)\right) \sum_{z=n+1}^{C_{au}} p_{au}(\underline{\alpha}_{au}, z) \\
&\cdot \prod_{\substack{y \in A^{-}_{aw}(ab) \\ y \neq w,d}} \left(1 - \left[\sum_{s=n}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=n}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \\
&\cdot \prod_{\substack{y \in A^{+}_{aw}(ab) \\ y \neq w,d,u}} \left(1 - \left[\sum_{s=n+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right] \left[\sum_{z=n+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \Bigg\}.
\end{aligned}
$$

The next partial derivatives we need to get are those of equation (3.22), and proceeding as for equation (3.21) we get the following expressions. Assume that OD pair $[a, w]$ has as first choice route the single link (a, w), also the link (a, b) is part of a two-link alternate route for this OD pair and is in state n, then obtain $\frac{\partial P_{aw}(w,b,n)}{\partial p_{av}(x)}$, $\forall v : (a, v) \in \mathcal{L}$, $v \neq b$ and $x = 0, 1, \ldots, C_{av}$ as follows

$$\frac{\partial P_{aw}(w,b,n)}{\partial p_{aw}(x)} = \begin{cases} \frac{P_{aw}(w,b,n)}{p_{aw}(\underline{\alpha}_{aw},0)}, & x = 0, \\ 0, \quad \text{otherwise.} \end{cases} \tag{4.92}$$

Consider that $n \geq x > 0$ then we have

$$\begin{aligned} \frac{\partial P_{aw}(w,b,n)}{\partial p_{wb}(x)} &= p_{aw}(0) \prod_{\substack{u \in A^-_{aw}(ab) \\ u \neq w}} \left(1 - \left[\sum_{s=x}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{r=x}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, r)\right]\right) \\ &\cdot \prod_{\substack{u \in A^+_{aw}(ab) \\ u \neq w}} \left(1 - \left[\sum_{s=x+1}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{r=x+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, r)\right]\right) \end{aligned} \tag{4.93}$$

Now for the case $x > n > 0$ we have

$$\begin{aligned} \frac{\partial P_{aw}(w,b,n)}{\partial p_{wb}(x)} &= p_{aw}(0) \prod_{\substack{u \in A^-_{aw}(ab) \\ u \neq w}} \left(1 - \left[\sum_{s=n}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{r=n}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, r)\right]\right) \\ &\cdot \prod_{\substack{u \in A^+_{aw}(ab) \\ u \neq w}} \left(1 - \left[\sum_{s=n+1}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{r=n+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, r)\right]\right) \end{aligned} \tag{4.94}$$

The following are the partial derivatives of the same probability $P_{aw}(w, b, n)$, but with respect to the probability distributions for the states of those links that form a two-link alternate route for OD pair $[a, w]$, i.e., links of the form $(a, u) \in \mathcal{L}$, $u \in \mathcal{T}_{aw}$, $u \neq b, w$, and links of the form $(u, w) \in \mathcal{L}$, $u \neq a, b$. First consider that link (a, u) is in state x and with link (a, b) in state n, then we have for $n > x > 0$

$$\begin{aligned} \frac{\partial P_{aw}(w,b,n)}{\partial p_{au}(x)} &= - \; p_{aw}(0) \Bigg\{ \mathcal{I}_{\{u \in A^-_{aw}(ab)\}} \sum_{r=1}^{x} \Bigg[p_{wb}(\underline{\alpha}_{wb}, r) \sum_{z=r}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw}, z) \\ &\cdot \prod_{\substack{y \in A^-_{aw}(ab) \\ y \neq w,u}} \left(1 - \left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay}, s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw}, z)\right]\right) \end{aligned} \tag{4.95}$$

$$
\cdot \prod_{\substack{y\in A^+_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]
$$

$$
+ \; \mathcal{I}_{\{u\in A^+_{aw}(ab)\}} \sum_{r=1}^{x-1}\Bigg[p_{wb}(\underline{\alpha}_{wb},r) \sum_{z=r+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)
$$

$$
\cdot \prod_{\substack{y\in A^-_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)
$$

$$
\cdot \prod_{\substack{y\in A^+_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]\Bigg\}.
$$

Now for the same case $n > x$, but for the partial derivative with respect to link (u, w) in state x, we have

$$
\frac{\partial P_{aw}(w,b,n)}{\partial p_{uw}(x)} = - \; p_{aw}(0)\Bigg\{\mathcal{I}_{\{u\in A^-_{aw}(ab)\}} \sum_{r=1}^{x}\Bigg[p_{wb}(\underline{\alpha}_{wb},r) \sum_{z=r}^{C_{au}} p_{au}(\underline{\alpha}_{au},z) \tag{4.96}
$$

$$
\cdot \prod_{\substack{y\in A^-_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)
$$

$$
\cdot \prod_{\substack{y\in A^+_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]
$$

$$
+ \; \mathcal{I}_{\{u\in A^+_{aw}(ab)\}} \sum_{r=1}^{x-1}\Bigg[p_{wb}(\underline{\alpha}_{wb},r) \sum_{z=r+1}^{C_{au}} p_{au}(\underline{\alpha}_{au},z)
$$

$$
\cdot \prod_{\substack{y\in A^-_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)
$$

$$
\cdot \prod_{\substack{y\in A^+_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]\Bigg\}.
$$

Next, the same partial derivative with respect to the same two links, (a, u) and (u, w), but the state x satisfies the condition $x \geq n > 0$, then we have

$$
\frac{\partial P_{aw}(w,b,n)}{\partial p_{au}(x)} = - \; p_{aw}(0)\Bigg\{\mathcal{I}_{\{u\in A^-_{aw}(ab)\}} \sum_{r=1}^{n}\Bigg[p_{wb}(\underline{\alpha}_{wb},r) \sum_{z=r}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z) \tag{4.97}
$$

$$
\cdot \prod_{\substack{y\in A^-_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)
$$

$$\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]$$

$$+ \ \mathcal{I}_{\{u\in A^{-}_{aw}(ab)\}} \sum_{r=n+1}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb},r) \sum_{z=n}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)$$

$$\cdot \prod_{\substack{y\in A^{-}_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=n}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=n}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)$$

$$\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=n+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=n+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)$$

$$+ \ \mathcal{I}_{\{u\in A^{+}_{aw}(ab)\}} \sum_{r=1}^{n-1}\Bigg[p_{wb}(\underline{\alpha}_{wb},r) \sum_{z=r+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)$$

$$\cdot \prod_{\substack{y\in A^{-}_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)$$

$$\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]$$

$$+ \ \mathcal{I}_{\{u\in A^{+}_{aw}(ab),\ x>n\}} \sum_{r=n}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb},r) \sum_{z=n+1}^{C_{uw}} p_{uw}(\underline{\alpha}_{uw},z)$$

$$\cdot \prod_{\substack{y\in A^{-}_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=n}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=n}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)$$

$$\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=n+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=n+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg\},$$

$$\frac{\partial P_{aw}(w,b,n)}{\partial p_{uw}(x)} = - \ p_{aw}(0)\Bigg\{\mathcal{I}_{\{u\in A^{-}_{aw}(ab)\}} \sum_{r=1}^{n}\Bigg[p_{wb}(\underline{\alpha}_{wb},r) \sum_{z=r}^{C_{au}} p_{au}(\underline{\alpha}_{au},z) \tag{4.98}$$

$$\cdot \prod_{\substack{y\in A^{-}_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)$$

$$\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\Bigg]$$

$$+ \ \mathcal{I}_{\{u\in A^{-}_{aw}(ab)\}} \sum_{r=n+1}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb},r) \sum_{z=n}^{C_{au}} p_{au}(\underline{\alpha}_{au},z)$$

$$\cdot \prod_{\substack{y\in A^{-}_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=n}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=n}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)$$

$$\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=n+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=n+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)$$

$$+ \ \mathcal{I}_{\{u\in A^{+}_{aw}(ab)\}} \sum_{r=1}^{n-1}\left[p_{wb}(\underline{\alpha}_{wb},r)\sum_{z=r+1}^{C_{au}} p_{au}(\underline{\alpha}_{au},z)\right.$$

$$\cdot \prod_{\substack{y\in A^{-}_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=r}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)$$

$$\left.\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=r+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=r+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\right]$$

$$+ \ \mathcal{I}_{\{u\in A^{+}_{aw}(ab),\ x>n\}} \sum_{r=n}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb},r) \sum_{z=n+1}^{C_{au}} p_{au}(\underline{\alpha}_{au},z)$$

$$\cdot \prod_{\substack{y\in A^{-}_{aw}(ab)\\ y\neq w}} \left(1-\left[\sum_{s=n}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=n}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)$$

$$\left.\cdot \prod_{\substack{y\in A^{+}_{aw}(ab)\\ y\neq w,u}} \left(1-\left[\sum_{s=n+1}^{C_{ay}} p_{ay}(\underline{\alpha}_{ay},s)\right]\left[\sum_{z=n+1}^{C_{yw}} p_{yw}(\underline{\alpha}_{yw},z)\right]\right)\right\}.$$

To finish the computation of the shadow prices we need the partial derivatives for equation (3.23). Assume that OD pair $[a,b]$ has as first-choice route the two-link path (a,d), (d,b), $d\neq b$ and the link (a,b) is in state n, then from (3.23) we get the following partial derivatives.

$$\frac{\partial P_{ab}(d,b,n)}{\partial p_{ad}(0)} = \frac{[1-p_{db}(\underline{\alpha}_{db},0)]P_{ab}(d,b,n)}{1-[1-p_{ad}(\underline{\alpha}_{ad},0)][1-p_{db}(\underline{\alpha}_{db},0)]}, \tag{4.99}$$

$$\frac{\partial P_{ab}(d,b,n)}{\partial p_{db}(0)} = \frac{[1-p_{ad}(\underline{\alpha}_{ad},0)]P_{ab}(d,b,n)}{1-[1-p_{ad}(\underline{\alpha}_{ad},0)][1-p_{db}(\underline{\alpha}_{db},0)]}. \tag{4.100}$$

Now consider the link (a,w) in state x, where $w\neq b,d$, then we get for $x\geq n>0$

$$\frac{\partial P_{ab}(d,b,n)}{\partial p_{aw}(x)} = - \ \left\{1-\left[1-p_{ad}(\underline{\alpha}_{ad},0)\right]\left[1-p_{db}(\underline{\alpha}_{db},0)\right]\right\} \tag{4.101}$$

$$
\begin{aligned}
&\Bigg\{ \mathcal{I}_{\{w\in A^-_{ab}(ab)\}} \sum_{r=n}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb}, r) \\
&\cdot \prod_{\substack{u\in A^-_{ab}(ab)\\ u\neq w,d,b}} \left(1-\left[\sum_{s=n}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{z=n}^{C_{ub}} p_{ub}(\underline{\alpha}_{ub}, z)\right]\right) \\
&\cdot \prod_{\substack{u\in A^+_{aw}(ab)\\ u\neq b,d}} \left(1-\left[\sum_{s=n+1}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{z=n+1}^{C_{ub}} p_{ub}(\underline{\alpha}_{ub}, z)\right]\right) \\
&+ \mathcal{I}_{\{w\in A^+_{ab}(ab),\ x>n\}} \sum_{r=n+1}^{C_{wb}} p_{wb}(\underline{\alpha}_{wb}, r) \\
&\cdot \prod_{\substack{u\in A^-_{ab}(ab)\\ u\neq b,d}} \left(1-\left[\sum_{s=n}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{z=n}^{C_{ub}} p_{ub}(\underline{\alpha}_{ub}, z)\right]\right) \\
&\cdot \prod_{\substack{u\in A^+_{ab}(ab)\\ u\neq b,w,d}} \left(1-\left[\sum_{s=n+1}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{z=n+1}^{C_{ub}} p_{ub}(\underline{\alpha}_{ub}, z)\right]\right)\Bigg]\Bigg\},
\end{aligned}
$$

and for $n > x > 0$, $\frac{\partial P_{ab}(d,b,n)}{\partial p_{aw}(x)} = 0$. Now, the same partial derivative, but with respect to the distribution of link (w, b) in state x, we get for $x \geq n > 0$

$$
\begin{aligned}
\frac{\partial P_{ab}(d,b,n)}{\partial p_{wb}(x)} = &- \Big\{1-\big[1-p_{ad}(\underline{\alpha}_{ad}, 0)\big]\big[1-p_{db}(\underline{\alpha}_{db}, 0)\big]\Big\} \\
&\Bigg\{ \mathcal{I}_{\{w\in A^-_{ab}(ab)\}} \sum_{r=n}^{C_{aw}} p_{aw}(\underline{\alpha}_{aw}, r) \\
&\cdot \prod_{\substack{u\in A^-_{ab}(ab)\\ u\neq w,d,b}} \left(1-\left[\sum_{s=n}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{z=n}^{C_{ub}} p_{ub}(\underline{\alpha}_{ub}, z)\right]\right) \\
&\cdot \prod_{\substack{u\in A^+_{aw}(ab)\\ u\neq b,d}} \left(1-\left[\sum_{s=n+1}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{z=n+1}^{C_{ub}} p_{ub}(\underline{\alpha}_{ub}, z)\right]\right) \\
&+ \mathcal{I}_{\{w\in A^+_{ab}(ab),\ x>n\}} \sum_{r=n+1}^{C_{aw}} p_{aw}(\underline{\alpha}_{aw}, r) \\
&\cdot \prod_{\substack{u\in A^-_{ab}(ab)\\ u\neq b,d}} \left(1-\left[\sum_{s=n}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{z=n}^{C_{ub}} p_{ub}(\underline{\alpha}_{ub}, z)\right]\right) \\
&\cdot \prod_{\substack{u\in A^+_{ab}(ab)\\ u\neq b,w,d}} \left(1-\left[\sum_{s=n+1}^{C_{au}} p_{au}(\underline{\alpha}_{au}, s)\right]\left[\sum_{z=n+1}^{C_{ub}} p_{ub}(\underline{\alpha}_{ub}, z)\right]\right)\Bigg]\Bigg\}.
\end{aligned}
\tag{4.102}
$$

The shadow prices are obtained substituting equations (4.82) through (4.103) in equation (4.80), taking this together with equations (4.77), (4.78) and (4.79) into (4.76), and the result together with equations (4.74) and (4.75) to be substituted in equation (4.73) where we get a set of simultaneous linear equations for $dp_{ab}(\underline{\alpha}_{ab}, m)/d\beta_{jk}(u)$. Solving this, we can find the values of $dB_{rs}(\underline{p})/d\beta_{jk}(u)$ in (4.70) by using (4.71) and (4.72) and use (4.68) to get the shadow price of the network rate of return.

4.6 Complexity

In addition, in this Chapter, we investigate the complexity of the fixed point and shadow price algorithm of LLR for a fully connected network with N nodes with every link in the network having a capacity of C circuits. Given the offered traffic to every link in every state, the complexity, i.e., the number of multiplications and additions to calculate the distribution for the states of every link is $O(CN^2)$, [8]. The complexity to calculate the offered traffic to every link is $O(CN^4)$ [8]. Now, let I be the number of iterations needed to obtain convergence in the repeated substitution for the solution of the fixed point. Then from [8] it follows that the complexity of calculating the stationary distribution of the states for all the links is $O(ICN^4)$. The calculation of blocking probability for all OD pairs has a complexity of $O(N^3)$.

For the shadow price algorithm, for LLR, the set of $(C+1)N(N-1)/2$ linear simultaneous equations in (4.24) needs to be solved. By solving the matrix equation using Coppersmith and Winograd's algorithm [47], which has complexity $O(n^{2.376})$ for a matrix of size n, it can be seen that the solution of this set of simultaneous equations has complexity $O(C^{2.376}N^{4.752})$. Equation (4.4) requires $O(C^2N^4)$ and (4.2) requires $O(N)$ operations.

Thus the algorithm for the evaluation of shadow prices using fast methods for matrix inversion has complexity $O(C^{2.376}N^{4.752})$. The algorithm for the evaluation of shadow prices for ALBA(K) has the same order of complexity.

In the next chapter, the shadow prices are used to find the sum capacity of the routing schemes and compare their performance by solving the constrained nonlinear optimization problem formulated.

Chapter 5

Application of Shadow Prices

The utility of the shadow prices for pricing policy seems obvious. At an operating point with established traffic levels, knowing the shadow prices of the network rate of return with respect to each of the exogenous arrival rates can allow the network manager to provision of penalties/discounts in order to improve the marginal rewards of additional traffic to the network. Table 5.1 lists the shadow prices at an operating point (from Figure 4.7) for the 4-node network using LLR and ALBA(4). From this table, it can be seen that to improve the marginal rewards to the network in the case of LLR, traffic in OD pairs $[0,1]$ and $[2,3]$ should be encouraged more than in OD pairs $[0,2]$, $[1,2]$ and $[1,3]$. In the case of ALBA(4), traffic in OD pairs $[0,1]$ and $[0,3]$ should be encouraged more than traffic in OD pairs $[0,2]$, $[1,2]$ and $[1,3]$.

5.1 Calculation of Sum Capacity

The sum capacity of a given adaptive routing scheme for a given network topology is defined as the largest sum of exogenous arrival rates that the given network topology can accommodate while maintaining a prescribed blocking probability for every origin-destination pair. This sum capacity is calculated by using the shadow prices in the solution of an appropriate constrained optimization problem. From the shadow prices of both LLR and ALBA(K) (i.e., ALBA with K generalized states for each link) we are able to calculate the sum capacities and compare these two closely related schemes in terms of the call carrying capacities they result in.

Table 5.1: Shadow Prices at an Operating Point

Four-Node Asymmetric Network			
OD Pair $[i,j]$	λ_{ij}	$dW/d\lambda_{ij}$ LLR	$dW/d\lambda_{ij}$ ALBA(4)
$[0,1]$	6.00	0.9231	0.8632
$[0,2]$	2.50	0.8360	0.6476
$[0,3]$	3.50	0.8891	0.8274
$[1,2]$	5.40	0.8045	0.5108
$[1,3]$	4.00	0.8678	0.7443
$[2,3]$	6.00	0.8945	0.7905

The sum capacity calculations indicate that even with adaptive routing substantial gains in network capacity are possible if the traffic is matched to the network topology. This is somewhat counterintuitive since, for example, in a fully connected network using an adaptive routing scheme such as LLR, each origin-destination pair is allowed to examine a large number of routes distributed over a substantial portion of the network while making a choice of route.

Shadow prices capture the effect of increases in external traffic in one OD pair on the *entire* network. As a result, they are useful in optimizing network-wide goals. Define the *sum capacity* of LLR and ALBA in asymmetric networks as the maximum sum of exogenous arrival rates such that the blocking probability of each OD pair is less than or equal to some prespecified maximum blocking probability. As an illustration of the use of shadow prices in optimizing network-wide goals, we use them to calculate the sum capacity. To this end, we formulate a constrained nonlinear optimization problem with the objective function being the rate of return and constraints being the blocking probabilities. The independent variables are the exogenous arrival rates. Let $\underline{\eta}$ be a vector whose components represent the maximum blocking probability for each OD pair and let $\underline{0}$ be the zero vector. Then the optimization problem is:

$$\begin{aligned} \max_{\underline{\lambda}} \quad W(\underline{\lambda}, \underline{\mathcal{B}}) &= \sum_{[j,k]\in\mathcal{O}} \lambda_{jk} w_{jk} \left(1 - B_{jk}(\underline{p})\right), \\ \text{subject to} \quad & \underline{\mathcal{B}} \leq \underline{\eta}, \\ & \underline{\lambda} \geq \underline{0}. \end{aligned} \tag{5.1}$$

The solution for the above optimization problem gives the maximum traffic that the network can carry for a given blocking probability vector. The optimization is achieved by using the shadow prices in a gradient descent algorithm that gives the direction in which the vector of exogenous arrival rates has to be varied to get the desired maximization. The specific algorithm used was the variable metric method [62] using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update formula. The step size is obtained by doing a line minimization of Powell's penalty function [62] and the algorithm is stopped when the improvement in the objective function $W(\underline{\lambda}, \underline{\mathcal{B}})$ is less than 10^{-7} and each blocking probability, $\mathcal{B}_i$, $1 \leq i \leq ||\mathcal{O}||$ (where $||\mathcal{O}||$ denotes the cardinality of $\mathcal{O}$) is in the interval, $(\eta_i - 10^{-4}, \eta_i)$ and the change in each of the decision variables λ_{ij}, $\forall\ [i,j] \in \mathcal{O}$ is no greater than 10^{-4}.

It is not known whether the objective function in the above optimization is concave and so it may be possible for the gradient descent algorithm to converge to local minima. In order to ensure that this did not occur we verified the result of the optimization using simulated annealing by starting at one of the solutions obtained by our algorithm and perturbing the decision variables [1]. In our examples there were no improvements obtained from simulated annealing.

In Figures 5.1 and 5.2, we plot the sum capacities for the 4-node and 5-node network, respectively, against the prescribed blocking probability which is the same for every OD pair. This allows us to compare different adaptive routing schemes. The figures also contain bounds on the sum capacity derived from the max-flow and Erlang bounds of [18] and the single-parented bound of [17]. To indicate the amount of improvement occuring from the optimization, the figures also provide the sum of exogenous rates for LLR and LLR with trunk reservation. This is obtained by keeping the exogenous arrival rates to all but one of the OD pairs constant and increasing that arrival rate until the network blocking reaches a particular value. In the figures, "optimized" refers to points obtained by solving the optimization problem (5.1). (Note that in these cases since all the OD pairs have a prespecified blocking probability, network blocking probability is equal to this OD pair blocking probability.)

Figures 5.1 and 5.2 show that using this optimization technique, for asymmetric networks using adaptive routing schemes such as LLR, significant improvements can be achieved in the sum of the external rates that the network can accommodate at a prescribed blocking. Clearly this improvement is possible if the external arrival rates are matched to the network topology. Another important observation that can be made from these figures is that ALBA(4) has about the same sum capacity as that of LLR with trunk reservation, $T = 2$, in fact usually a little better, hence reservation levels have to be chosen wisely to improve performance.

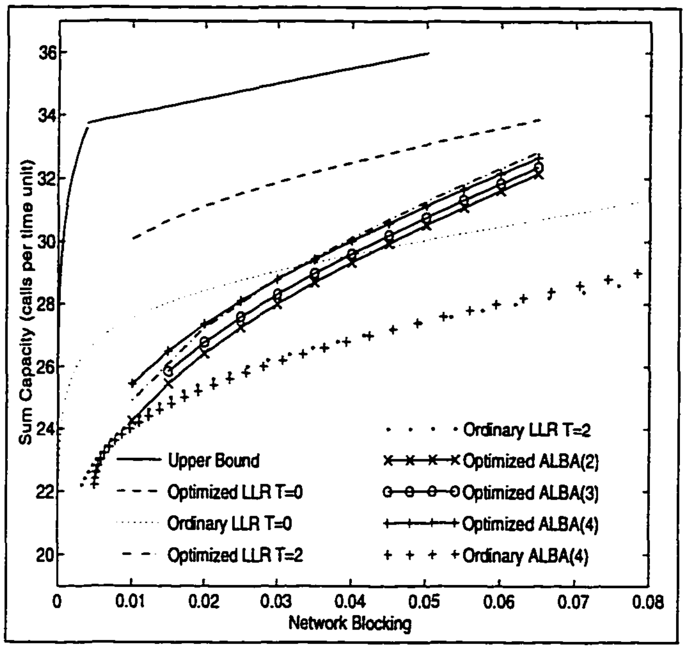

Figure 5.1: Sum Capacity for a Four-Node Asymmetric Network

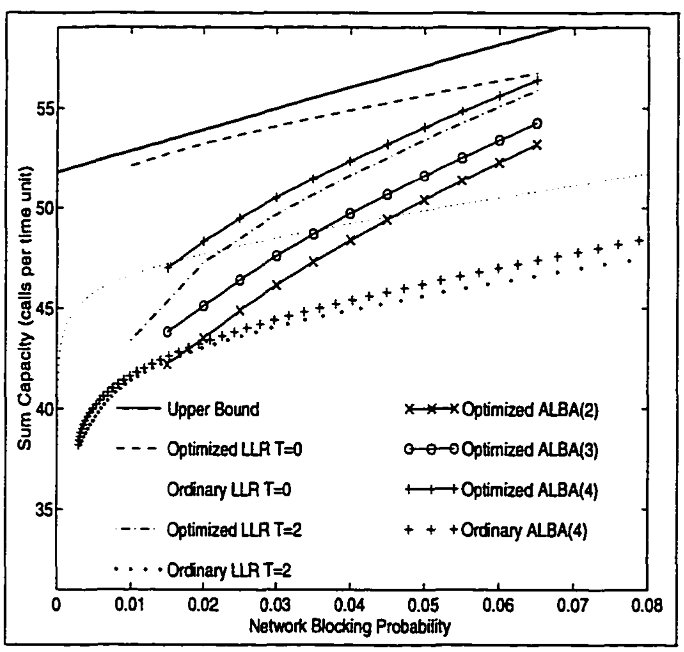

Figure 5.2: Sum Capacity for a Five-Node Asymmetric Network

We also investigated the sum capacity of symmetric networks with 4 and 5 nodes with the capacity of each link chosen to be the average capacity of the corresponding asymmetric network. Figures 5.3 and 5.4 show the sum capacity, and bounds for symmetric fully-connected networks with 4 and 5 nodes, respectively. As mentioned before, the capacity of each link was chosen to be the average capacity of the asymmetric network. Comparing these figures with Figures 5.1 and 5.2, we see that in almost all the cases the optimized symmetric networks have a sum capacity which is not greatly different from the sum capacity of the corresponding optimized asymmetric network. This is remarkable because there is a wide variation in the link capacities of the asymmetric network and the symmetric network has a *total* capacity of all links equal to that of the asymmetric network. This indicates that the characteristics of the adaptive routing schemes wherein they consider state information from the entire network for routing and are allowed a number of routes in the routing set of each OD pair allow the entire network to be considered as one resource from the point of view of routing.

This serves to confirm the appropriateness of sum capacity as a measure of network capacity. Of interest also are the different ways in which the various adaptive routing schemes behave in the cases of symmetric and asymmetric networks. For instance, for the case of LLR with no trunk reservation, for all the networks considered, the symmetric case yields a larger sum capacity than the asymmetric case for almost all blocking probabilities of interest. For the case of optimized ALBA(3) however, the asymmetric case yields the larger sum capacity. The graphs reveal

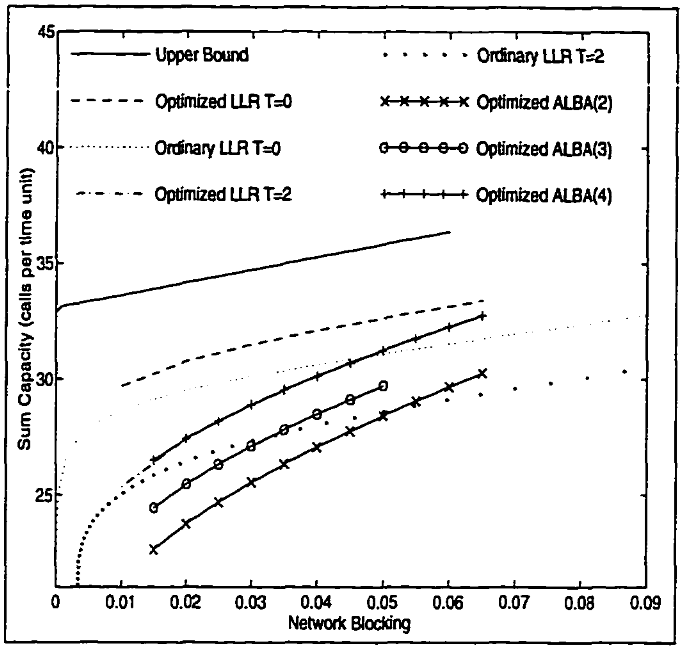

Figure 5.3: Sum Capacity for a Four-Node Symmetric Network

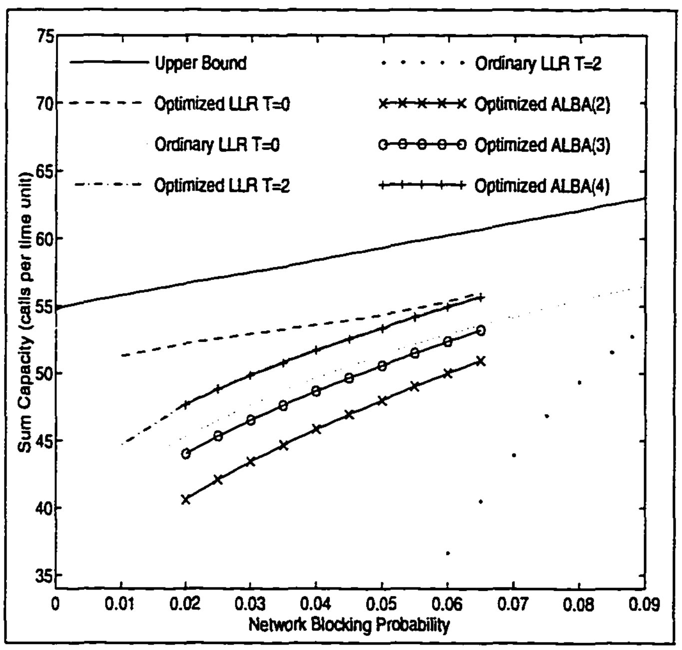

Figure 5.4: Sum Capacity for a Five-Node Symmetric Network

several other instances of such anomalous behavior. The use of shadow prices thus allows to make quantifiable assessments of the effects of asymmetry in the network for various adaptive routing schemes and suggests that this be another basis for performance comparison between these routing schemes.

In figures 5.5 and 5.6, the comparison of RALBA(2) and RALBA(3) to LLR with and without trunk reservation by the sum capacity criterion is presented. It can be seen that LLR performs better than RALBA for all the cases analyzed.

5.2 Numerical Results for MLLR

The numerical results obtained for MLLR were for fully connected networks with four and five nodes, the capacity of the links and external arrival rates per OD pair are shown in tables 5.2, 5.3 and 5.4, the external arrival rate for each OD pair was chosen to be 75 percent of the capacity of its direct link and for the OD pair with least capacity in its single-link path the exogenous traffic was varied from zero to 15 calls per time unit at intervals of 0.25 and 0.5, to obtain the performance comparison. The results presented include the performance evaluation through the OD pair blocking probability with the parameters β_{ij} fixed and obtained by the constrained optimization problem formulated later and the rate of return from the network.

5.2.1 Sum Capacity

We used shadow prices to get the sum capacity of LLR and ALBA in asymmetric networks, in [67] i.e., the maximum external arrival rates such that the blocking probability of each OD pair is less than or equal to some given maximum OD pair

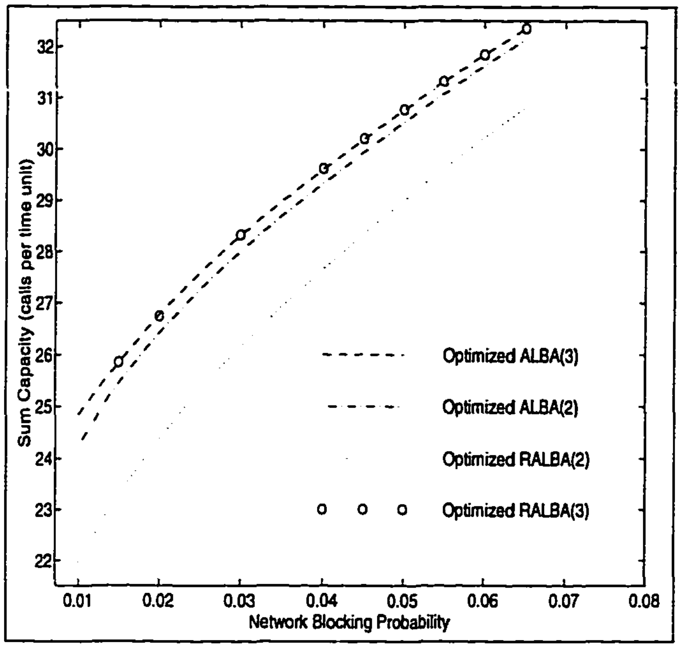

Figure 5.5: Sum Capacity for a Four-Node Network using ALBA and RALBA

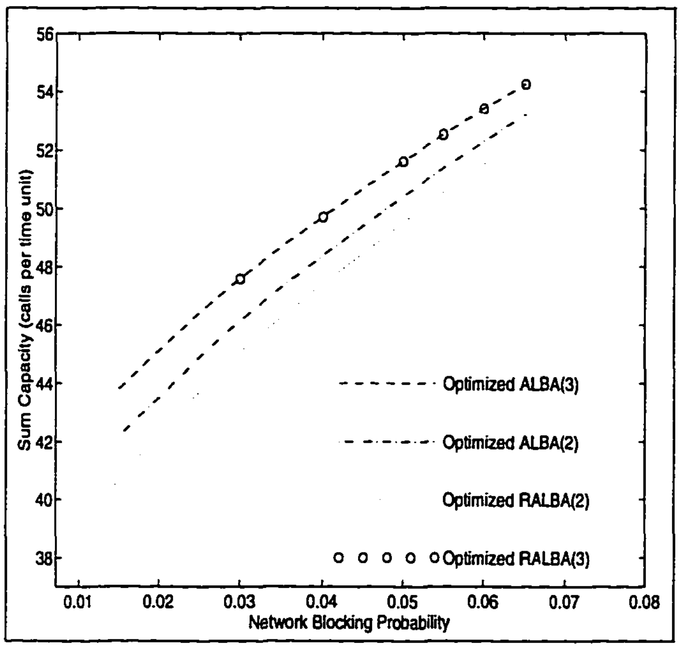

Figure 5.6: Sum Capacity for Five-Node Network using ALBA and RALBA

blocking probability. Due to the lack of control over the external arrival rates in the network, we modified the optimization problem for this routing scheme, where we use shadow prices to determine the set of β_{ij}'s such that the traffic carried is maximum for a given set of external arrival rates per OD pair.

To this end, define a constrained nonlinear optimization problem with objective function being the rate of return and constraints being the distributions β_{ij}'s. The independent variables are these load sharing coefficients. Then the optimization problem formulated is:

$$\begin{aligned}
\max_{\beta_{ij},\forall[i,j]\in\mathcal{O}} \quad W(\underline{B}) &= \sum_{[j,k]\in\mathcal{O}} \lambda_{jk} w_{jk}\left(1 - B_{jk}(\underline{p})\right), \\
\text{subject to} \quad & \sum_{u\in S_{ij}} \beta_{ij}(u) = 1, \quad \forall[i,j]\in\mathcal{O} \\
& 0 \le \beta_{ij}(u) \le 1 \quad \forall[i,j]\in\mathcal{O}, \quad \forall u\in\mathcal{Q}_{ij}. \qquad (5.2)
\end{aligned}$$

The solution for the above optimization problem gives the maximum traffic that the network can carry for a given external arrival rate vector. The optimization is achieved by using the shadow prices in a gradient descent algorithm that gives the direction in which the vector of distributions β_{ij}'s has to be varied in order to maximize the network rate of return.

5.2.2 Performance Comparison

Every OD pair considered in the Network has a set of available routes consisting of the single-link and two-link paths. The capacity available in each of the routes for the two-link paths is the minimum of the number of free circuits in each link. Although the load sharing coefficients, $\beta_{ij}(u)$, can be related to the capacity available in each of the routes in some way, i.e., some percent of total available capacity or proportional to this in each OD pair, the optimization problem solved will give the exact coefficients that maximize the carried traffic, and as it will be seen, it depends on the asymmetry of the network. The motivation to solve this problem comes from the idea of considering an infinite number of cases given by different values of the coefficients and finding the optimal, where LLR is one of the cases.

Table 5.2 contains the capacities for the links of the 4-Node network used in the first example. In this case, the optimization algorithm gave as a result the load sharing coefficients corresponding to LLR, i.e., $\beta_{ij} = (1,0,0)$, $\forall[i,j] \in \mathcal{O}$. Related results to this case involving performance evaluation and maximization of traffic can be found in [67]. The result given by this optimization is not surprising since the network is not highly asymmetric and the capacity available in each route for every OD pair is about the same. Hence, in this example, optimized MLLR and LLR perform equally.

In Table 5.3 the 4-Node network is highly asymmetric. The results obtained by the optimization for the coefficients were the same as LLR for high input traffic to OD pair $[0,1]$, in the order of five units or more, but for values of external arrival rate less than five units, we get that the coefficients are those of LLR for all the OD pairs except OD pair $[0,1]$, which gets the distribution $\beta_{01} = (0,1,0)$. This can be related to the maximum capacity available in each of the routes for this OD pair, since for the second route, i.e., path $(0,2)$, $(2,1)$, it is 25, for the first route, i.e., single-link, is 2 and for the third route, i.e., path $(0,3)$, $(3,1)$, is 20. The performance evaluation is in Figure 5.7, where it is seen that for input traffic below five units,

Table 5.2: Four-Node Fully Connected Network. (*λ_{12} was varied)

Network Description		
OD Pair $[i,j]$	C_{ij}	λ_{ij}
$[0,1]$	12	9.00
$[0,2]$	5	3.75
$[0,3]$	7	5.25
$[1,2]$	4	*
$[1,3]$	8	6.00
$[2,3]$	12	9.00

Table 5.3: Four-Node Fully Connected Network. (*λ_{01} was varied)

Network Description		
OD Pair $[i,j]$	C_{ij}	λ_{ij}
$[0,1]$	2	*
$[0,2]$	50	37.50
$[0,3]$	20	15.00
$[1,2]$	25	18.75
$[1,3]$	40	30.00
$[2,3]$	60	45.00

MLLR performs better than LLR. In Figure 5.8 we have the rate of return for the input traffic given, and although LLR and MLLR are not much different, it can be seen that MLLR carries more traffic than LLR.

For the example of the network of Table 5.4, we have in Figure 5.9 the performance evaluation, where it is shown that making the network less asymmetric, we can affect the performance of MLLR which becomes closer to that of LLR. In the exogenous arrival rates where these curves differ, the load sharing coefficients for OD pair [1,2] are (0,1,0), which is consistent with the available capacity argument given in Example 2. The rate of return for this network is in Figure 5.10, where there is no significant difference in the carried traffic using both schemes.

Table 5.4: Four-Node Fully Connected Network. (* λ_{12} was varied)

Network Description		
OD Pair $[i, j]$	C_{ij}	λ_{ij}
[0,1]	2	1.50
[0,2]	40	30.00
[0,3]	7	5.25
[1,2]	1	*
[1,3]	30	22.50
[2,3]	50	37.50

The examples presented were for a fully connected network with four nodes, similar results can be obtained for networks with more nodes or with higher capacities but with similar characteristics concerning the asymmetry. In these results we did not consider trunk reservation parameters.

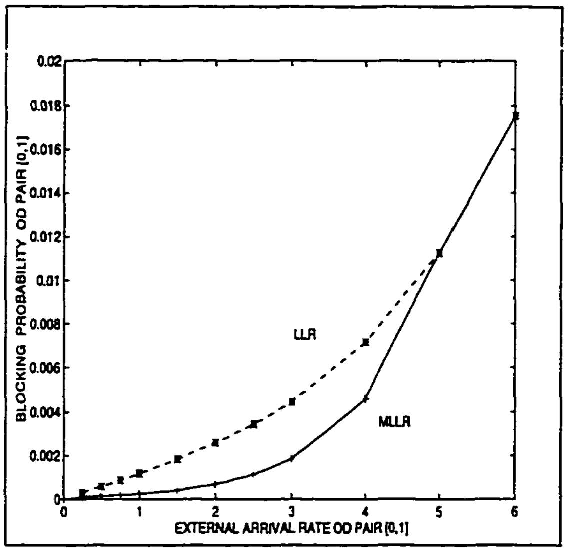

Figure 5.7: Blocking Probability OD pair [0, 1] for a Four-Node Network using LLR and MLLR

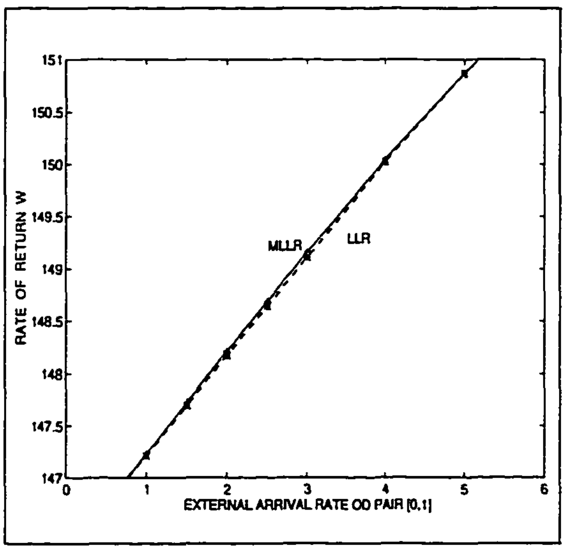

Figure 5.8: Rate of Return for a Four-Node Network using LLR and MLLR

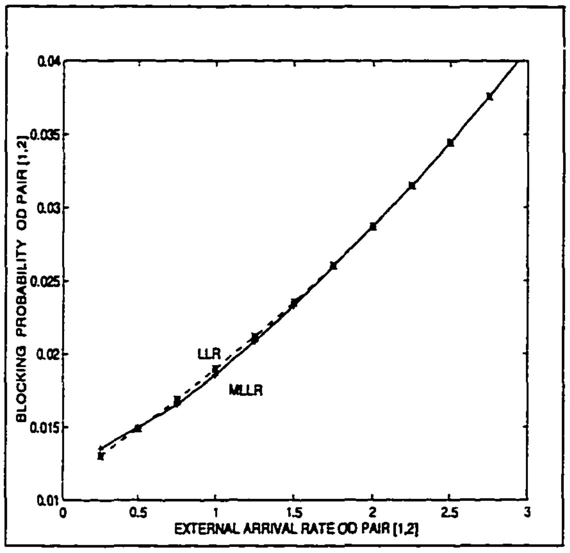

Figure 5.9: Blocking Probability OD pair [1, 2] for a Four-Node Network using LLR and MLLR

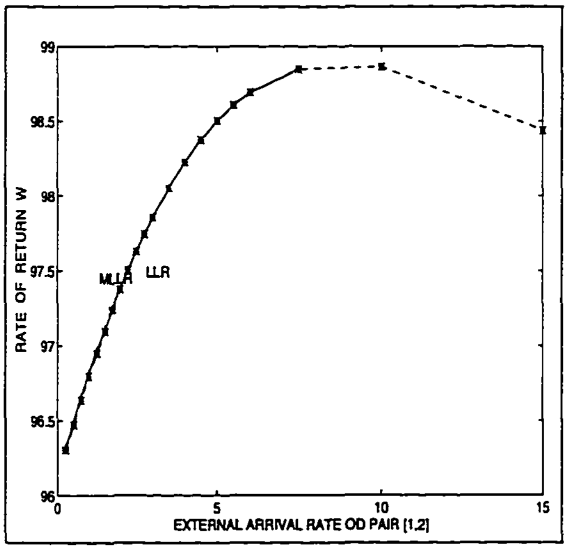

Figure 5.10: Rate of Return for a Four-Node Network using LLR and MLLR

The results obtained using shadow prices to maximize the network rate of return were motivating when we compare several routing schemes under the same framework. In the next chapters, we extend these calculations to consider networks with multiple classes of traffic and wireless networks. Networks with multiple classes of customers have been considered due to the importance of service integration in a media that could deliver with good quality of service information like voice, video and data. Wireless networks have recently become an important part of the telecommunication media used by the public.

Chapter 6

Multiservice Networks

Recently, networks have evolved to the point where transmission of different classes of information, such as video, voice and data, is needed. These classes have different characteristics and require different types of services from the network, i.e., different bandwidth requirements, holding times, arrival rates, quality of service, etc. For circuit-switched networks with multiple classes of traffic, there have been several approaches to this model, as those mentioned in Section 6.1, but we extend the model for single rate LLR of [8] to introduce Multirate LLR in Section 6.2 and give examples for the case of two classes of traffic.

Multirate traffic routing has been considered in [9], [24] and [14]. In all these cases, fixed routing was analyzed, whereas in this chapter, multirate adaptive routing which considers state dependent offered traffic to the links is presented.

This chapter also introduces the shadow price calculation for a network with multiple classes of traffic. The network model used and the characteristics of the traffic classes are explained in Section 6.2.1, some examples are presented of the performance evaluation through OD pair blocking probability where the routing policy used is LLR for each class. First, in Section 6.1, related work is presented, where a brief discussion on traffic behavior is introduced.

6.1 Background

It is known that the overflow traffic from a link with finite capacity offered Poisson arrivals is peaked, i.e., the variance to mean ratio of the overflow traffic is greater than one. For a treatment of traffic models that consider peaked overflows see [20].

In [24], the problem of peaked overflow traffic versus Poisson overflow is addressed. In that case, the network analyzed has two classes of traffic, one Narrow Band (NB) and one Wide Band (WB). The routing policies are that for NB calls, try the direct single-link route first, if there is free capacity, then connect the call, if there is no capacity available, then attempt a set of two-link alternate routes ordered in a sequence and connecting the call in the first one that is available. For the WB calls, they are routed only through the single-link, and if there is no sufficient capacity to accomodate them they join an infinite first-in first-out queue.

The overflow traffic offered to the alternate routes is taken to be generated by an Interrupted Poisson Process (IPP) where Poisson arrivals are turned on and off for random times exponentially distributed. The link model considers two different Markov chains, one for NB calls and one for WB calls coupled by some parameters. The link arrival rates are not state dependent, and the fixed point equations are solved to obtain the distribution of the number of NB calls given a number of WB calls in service, i.e., assuming that the NB call process reaches steady state before the number of WB calls changes. The performance measures were taken to be the OD pair blocking probability for NB calls and the average delay for WB calls. The main results are that with overflows assumed to be Poisson, the numerical evaluation gives a blocking probability which is conservative compared to that of simulation, and that using the IPP model it results in a better approximation. The examples included were for symmetric networks, although only numerical results were included for asymmetric networks, where the Poisson assumption had more discrepancy with respect to the IPP model as the number of alternate routes increased.

State dependent routing tends to smooth the overflow traffic offered to each link, making the Poisson assumption reasonable in the context of adaptive routing [49].

In [73], a performance evaluation algorithm for a network with two classes of traffic is presented, where the two classes have the same bandwidth requirements and different service times and arrival rates. The model is solved by using a two-dimensional Markov Chain with non-state dependent birth rates. Direct routing and alternate routing is considered as well as trunk reservation and priority control.

In [9], performance evaluation models are presented for a network with fixed routing. The models consider the offered traffic to a link as a reduced load approximation and the Knapsack approximation and Pascal approximation are included. At the end of the paper, shadow prices are introduced under the same framework as that in [35] extending the idea to consider multiple classes of traffic.

In [14], multirate fixed routing shadow prices and the maximization of the network rate of return are introduced. The optimization is performed using a two-phase procedure. In the first phase, a convex approximation scheme is used where a concave function, which is also a lower bound for the true network rate of return, is maximized globally and the result is set as the initial point of the true optimization in the second phase. The approach used to obtain the derivatives is an extension of that in [36].

In the model presented in the following section, we consider that overflow traffic is Poisson with state dependent offered load to each link and that for both classes of traffic we use LLR. Results of numerical evaluation are compared to those of simulation and the effect of trunk reservation in the performance is presented, where the approximation of the fixed point equations becomes more accurate as the reservation level increases.

6.2 Multirate LLR

The network will use LLR for each of the classes of traffic. The notation will follow definitions introduced in Section 3.1 as well as some new notation introduced in Section 6.2.1.

6.2.1 Model Description

Consider a fully connected network with N nodes, $\binom{N}{2}$ Origin Destination (OD) pairs and each link (j,k) with capacity C_{jk}. K classes of traffic will share the network resources. let b_i be the number of circuits required by a class i of traffic, $i = 1, 2, \ldots, K$ and without loss of generality consider $0 < b_1 < b_2 < \cdots < b_K$, let the service time be exponentially distributed and independent of previous service times with mean $1/\mu_i$ for traffic class i, let the arrival process for each OD pair be Poisson independent of other arrival processes with mean $\lambda_{jk}^{(i)}$ for traffic class i for OD pair $[j,k]$, let $p_{jk}(\mathrm{n})$ be the stationary probability that link (j,k) is in state

$\mathbf{n} = (n_1, n_2, \ldots, n_K)$ where n_i is the number of calls present in link (j,k) of class i. Define $\mathbf{n}^+ = (n_1, n_2, \ldots, n_i + 1, \ldots, n_K)$ and $\mathbf{n}^- = (n_1, n_2, \ldots, n_i - 1, \ldots, n_K)$, let $\alpha_{jk}^{(i)}(\mathbf{n})$ be the offered traffic of class i to link (j,k) when the link is in state $\mathbf{n}$ and $\alpha_{jk}^{T}(\mathbf{n})$ the total offered traffic to link (j,k) when the link is in state $\mathbf{n}$. Let T_i be the trunk reservation parameter for traffic class i, i.e., the number of calls reserved for direct routed traffic of class i. Denote the set of feasible states for link (j,k) as Ω_{jk}, i.e., the set of states $\mathbf{n}$ such that $\sum_{i=1}^{K} n_i b_i \leq C_{jk}$. Define the indicator function for the free capacity for traffic class m in link (j,k) as

$$f_{jk}^{(m)}(\mathbf{n}) = \begin{cases} 1, & \text{if } C_{jk} - \sum_{i=1}^{K} n_i b_i \geq b_m, \\ 0, & otherwise. \end{cases} \tag{6.1}$$

Equation (6.1) is the indicator function of the state $\mathbf{n}_m^+$ for link (j,k), i.e., $f_{jk}^{(m)}(\mathbf{n}) = 1$ if $\mathbf{n}_m^+ \in \Omega_{jk}$, it is zero otherwise; or it can also be seen as the indicator function of the *unblocked* states, i.e., those states for which at least one more call of class m can be accepted. Define $\mathcal{U}_{jk}^{(m)}$ as the set of *unreserved* states $\mathbf{n}$ for traffic class m in link (j,k) as follows

$$\mathcal{U}_{jk}^{(m)} = \left\{ \mathbf{n} : \; T_m < \left\lfloor \frac{C_{jk} - \sum_{i=1}^{K} n_i b_i}{b_m} \right\rfloor \right\}, \tag{6.2}$$

correspondingly, define the set of *reserved* states for traffic class m in link (j,k) as

$$\mathcal{Q}_{jk}^{(m)} = \left\{ \mathbf{n} : \; T_m \geq \left\lfloor \frac{C_{jk} - \sum_{i=1}^{K} n_i b_i}{b_m} \right\rfloor \right\}, \tag{6.3}$$

the set of *blocked* states for traffic class m in link (j,k) as

$$\mathcal{B}_{jk}^{(m)} = \left\{ \mathbf{n} : \; C_{jk} - \sum_{i=1}^{K} n_i b_i < b_m \right\}, \tag{6.4}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. This performance evaluation model is represented by a K-dimensional Markov chain with state dependent birth rates. In Figure 6.1 there is an example of a Markov chain when there are $K = 2$ classes of traffic. The set of reserved states for both classes of traffic is also shown,

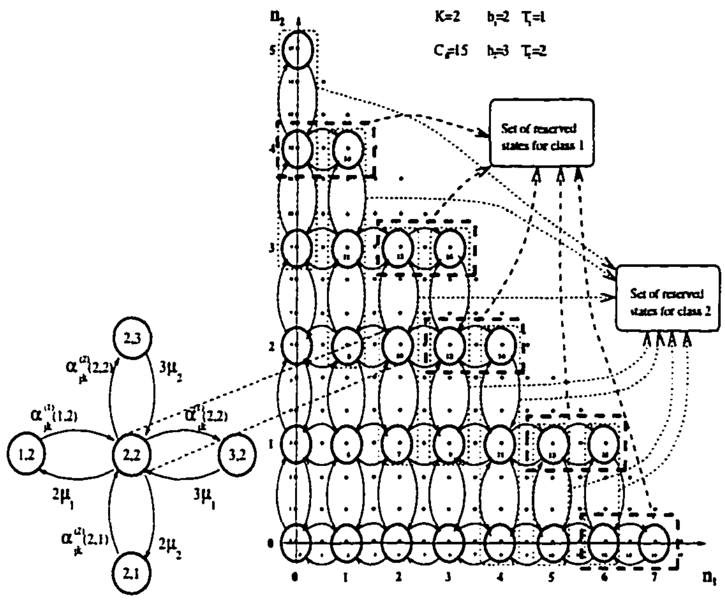

Figure 6.1: Markov Chain for Two Classes of Traffic on Link (j,k)

and the transition rates for state $(2,2)$.

Define $\mathcal{I}_{\{n_i>0\}}$ as one if $n_i > 0$ and zero otherwise. From the K-dimensional Markov chain we can obtain the global balance equations for link (j,k) in state $\mathbf{n}$ fixed as follows

$$\left\{\sum_{i=1}^{K} n_i\mu_i + \sum_{i=1}^{K} f_{jk}^{(i)}(\mathbf{n})\alpha_{jk}^{(i)}(\mathbf{n})\right\} p_{jk}(\mathbf{n}) = \sum_{i=1}^{K} \mathcal{I}_{\{n_i>0\}}\alpha_{jk}^{(i)}(\mathbf{n}_i^-)p_{jk}(\mathbf{n}_i^-) + \sum_{i=1}^{K} f_{jk}^{(i)}(n_i+1)\mu_i p_{jk}(\mathbf{n}_i^+), \quad (6.5)$$

where the distribution for the states of link (j,k) must satisfy

$$\sum_{\mathbf{n}\in\Omega_{jk}} p_{jk}(\mathbf{n}) = 1; \quad (6.6)$$

The total offered traffic to a link is the sum of the traffics offered by each of the classes to such link, i.e.,

$$\alpha_{jk}^{T}(\mathbf{n}) = \sum_{i=1}^{K} \alpha_{jk}^{(i)}(\mathbf{n}), \tag{6.7}$$

then we have for each traffic class i and for link (j,k) when this is in a blocked state

$$\alpha_{jk}^{(i)}(\mathbf{n}) = 0, \quad \text{if } \ \mathbf{n} \in \mathcal{B}_{jk}^{(i)}, \tag{6.8}$$

when the link is in the reserved states we have

$$\alpha_{jk}^{(i)}(\mathbf{n}) = \lambda_{jk}^{(i)}, \quad \text{if } \ \mathbf{n}_i^{+} \in \Omega_{jk} \ \text{ and } \ \mathbf{n} \in \mathcal{Q}_{jk}^{(i)} \backslash \mathcal{B}_{jk}^{(i)}. \tag{6.9}$$

The following definitions will be used to obtain the overflow traffic offered to a link from other OD pairs. Denote by $q_{jk}(i)$ the probability that link (j,k) cannot accept another class i call, i.e.,

$$q_{jk}(i) = \sum_{\mathbf{n} \in \mathcal{B}_{jk}^{(i)}} p_{jk}(\mathbf{n}). \tag{6.10}$$

Define for traffic class m and link (j,k) in state $\mathbf{n} \in \mathcal{U}_{jk}^{(m)}$ fixed, the set of states $\mathbf{r} \in \mathcal{U}_{uv}^{(m)}$ of link (u,v) such that the free capacity of link (u,v) is greater than or equal to that of link (j,k) as

$$\overline{\mathcal{M}_{jk}^{uv}}(m,\mathbf{r},\mathbf{n}) = \left\{ \mathbf{r} \in \mathcal{U}_{uv}^{(m)} : C_{jk} - \sum_{i=1}^{K} n_i b_i \leq C_{uv} - \sum_{i=1}^{K} r_i b_i \right\}, \tag{6.11}$$

and define also for fixed state $\mathbf{r} \in \mathcal{U}_{uv}^{(m)}$ of link (u,v) and traffic class m the set of states $\mathbf{n}$ of link (j,k) for which the free capacity of link (j,k) is smaller than or equal to that of link (u,v) as

$$\overline{\mathcal{P}_{jk}^{uv}}(m,\mathbf{n},\mathbf{r}) = \left\{ \mathbf{n} \in \mathcal{U}_{jk}^{(m)} : \ \mathbf{r} \in \mathcal{U}_{uv}^{(m)}, \ C_{jk} - \sum_{i=1}^{K} n_i b_i \leq C_{uv} - \sum_{i=1}^{K} r_i b_i \right\}, \tag{6.12}$$

in the same form, define $\mathcal{M}^{uv}_{jk}(m,\mathbf{r},\mathbf{n})$ and $\mathcal{P}^{uv}_{jk}(m,\mathbf{n},\mathbf{r})$ as in 6.11 and 6.12, respectively, but with strict inequality. Now, let $\mathbf{n}$ be an unreserved state, i.e., $\mathbf{n} \in \mathcal{U}^{(m)}_{jk}$, then we get for the offered traffic of class i to link (j,k) when this link is in state $\mathbf{n}$

$$\begin{aligned} \alpha^{(i)}_{jk}(\mathbf{n}) &= \sum_{v:[j,v]\in\mathcal{O}} \lambda_{jv}(i) q_{jv}(i) P^{(i)}_{jv}[(j,k),\mathbf{n}] \\ &+ \sum_{v:[v,k]\in\mathcal{O}} \lambda_{vk}(i) q_{vk}(i) P^{(i)}_{vk}[(j,k),\mathbf{n}] + \lambda_{jk}(i), \end{aligned} \tag{6.13}$$

where $P^{(i)}_{jv}[(j,k),\mathbf{n}]$ is the probability that OD pair $[j,v]$ chooses for traffic class i that two-link alternate route that uses link (j,k) when this is in state $\mathbf{n}$, respectively for $P^{(i)}_{vk}[(j,k),\mathbf{n}]$, and they are obtained as

$$\begin{aligned} P^{(i)}_{jv}[(j,k),\mathbf{n}] &= \sum_{\mathbf{m}\in\overline{\mathcal{P}^{jk}_{kv}}(i,\mathbf{m},\mathbf{n})} p_{kv}(\mathbf{m}) \\ &\cdot \prod_{d\in A^{-}_{jv}(jk)} \left\{1 - \left[\sum_{\mathbf{r}\in\overline{\mathcal{M}^{jd}_{kv}}(i,\mathbf{r},\mathbf{m})} p_{jd}(\mathbf{r})\right]\left[\sum_{\mathbf{x}\in\overline{\mathcal{M}^{dv}_{kv}}(i,\mathbf{x},\mathbf{m})} p_{dv}(\mathbf{x})\right]\right\} \\ &\cdot \prod_{d\in A^{+}_{jv}(jk)} \left\{1 - \left[\sum_{\mathbf{r}\in\mathcal{M}^{jd}_{kv}(i,\mathbf{r},\mathbf{m})} p_{jd}(\mathbf{r})\right]\left[\sum_{\mathbf{x}\in\mathcal{M}^{dv}_{kv}(i,\mathbf{x},\mathbf{m})} p_{dv}(\mathbf{x})\right]\right\} \\ &+ \sum_{\mathbf{m}\in\mathcal{M}^{kv}_{jk}(i,\mathbf{m},\mathbf{n})} p_{kv}(\mathbf{m}) \\ &\cdot \prod_{d\in A^{-}_{jv}(jk)} \left\{1 - \left[\sum_{\mathbf{r}\in\overline{\mathcal{M}^{jd}_{jk}}(i,\mathbf{r},\mathbf{n})} p_{jd}(\mathbf{r})\right]\left[\sum_{\mathbf{x}\in\overline{\mathcal{M}^{dv}_{jk}}(i,\mathbf{x},\mathbf{n})} p_{dv}(\mathbf{x})\right]\right\} \\ &\cdot \prod_{d\in A^{+}_{jv}(jk)} \left\{1 - \left[\sum_{\mathbf{r}\in\mathcal{M}^{jd}_{jk}(i,\mathbf{r},\mathbf{n})} p_{jd}(\mathbf{r})\right]\left[\sum_{\mathbf{x}\in\mathcal{M}^{dv}_{jk}(i,\mathbf{x},\mathbf{n})} p_{dv}(\mathbf{x})\right]\right\} \end{aligned} \tag{6.14}$$

Similarly, for the contribution from OD pair $[v,k]$ to the offered traffic to link (j,k) of class i we get the term $P^{(i)}_{vk}[(j,k),\mathbf{n}]$ which is obtained as

$$P^{(i)}_{vk}[(j,k),\mathbf{n}] = \sum_{\mathbf{m}\in\overline{\mathcal{P}^{jk}_{jv}}(i,\mathbf{m},\mathbf{n})} p_{jv}(\mathbf{m}) \tag{6.15}$$

$$
\begin{aligned}
&\cdot \prod_{d \in A^-_{vk}(jk)} \left\{ 1 - \left[\sum_{\mathbf{r} \in \overline{\mathcal{M}^{kd}_{jv}}(i,\mathbf{r},\mathbf{m})} p_{kd}(\mathbf{r}) \right] \left[\sum_{\mathbf{x} \in \overline{\mathcal{M}^{dv}_{jv}}(i,\mathbf{x},\mathbf{m})} p_{dv}(\mathbf{x}) \right] \right\} \\
&\cdot \prod_{d \in A^+_{vk}(jk)} \left\{ 1 - \left[\sum_{\mathbf{r} \in \mathcal{M}^{kd}_{jv}(i,\mathbf{r},\mathbf{m})} p_{kd}(\mathbf{r}) \right] \left[\sum_{\mathbf{x} \in \mathcal{M}^{dv}_{jv}(i,\mathbf{x},\mathbf{m})} p_{dv}(\mathbf{x}) \right] \right\} \\
+ &\sum_{\mathbf{m} \in \mathcal{M}^{jv}_{jk}(i,\mathbf{m},\mathbf{n})} p_{jv}(\mathbf{m}) \\
&\cdot \prod_{d \in A^-_{vk}(jk)} \left\{ 1 - \left[\sum_{\mathbf{r} \in \overline{\mathcal{M}^{kd}_{jk}}(i,\mathbf{r},\mathbf{n})} p_{kd}(\mathbf{r}) \right] \left[\sum_{\mathbf{x} \in \overline{\mathcal{M}^{dv}_{jk}}(i,\mathbf{x},\mathbf{n})} p_{dv}(\mathbf{x}) \right] \right\} \\
&\cdot \prod_{d \in A^+_{vk}(jk)} \left\{ 1 - \left[\sum_{\mathbf{r} \in \mathcal{M}^{kd}_{jk}(i,\mathbf{r},\mathbf{n})} p_{kd}(\mathbf{r}) \right] \left[\sum_{\mathbf{x} \in \mathcal{M}^{dv}_{jk}(i,\mathbf{x},\mathbf{n})} p_{dv}(\mathbf{x}) \right] \right\}
\end{aligned}
$$

The blocking probability of class i traffic for OD pair $[j,k]$, $B^{(i)}_{jk}$, is given by

$$
B^{(i)}_{jk} = q_{jk}(i) \prod_{d \in \mathcal{T}_{jk}} \left\{ 1 - \left[\sum_{\mathbf{r} \in \mathcal{U}^{(i)}_{jd}} p_{jd}(\mathbf{r}) \right] \left[\sum_{\mathbf{x} \in \mathcal{U}^{(i)}_{dk}} p_{dk}(\mathbf{x}) \right] \right\}. \tag{6.16}
$$

Table 6.1: 4-Node Network with Two Traffic Classes ($^*\lambda^{(1)}_{13}$ was varied)

Multirate LLR			
OD Pair $[i,j]$	C_{ij}	$\lambda^{(1)}_{ij}$	$\lambda^{(2)}_{ij}$
$[0,1]$	12	2.00	3.400
$[0,2]$	14	3.50	3.050
$[0,3]$	15	3.95	3.125
$[1,2]$	16	4.20	3.200
$[1,3]$	10	*	1.500
$[2,3]$	17	4.75	3.275

The comparison of the simulation results to numerical evaluation, gets better as the trunk reservation increases, as can be seen for the 4-node network with capacities and external arrival rates as shown in table 6.1 and service rate for the first traffic class equal to one and for the second equal to one half. Figures 6.2 and 6.3 show the OD pair [1, 3] blocking probability for traffic class 1 when $\lambda_{13}^{(1)}$ was varied and the error for all OD pairs between the approximation and simulation for this traffic class. It can be seen that this error is between ten and twenty percent. Figures 6.4 and 6.5 show the OD pair [1, 3] blocking probability for traffic class 2 and the error of all the OD pairs for this traffic class. It can be seen that the error is less than eight percent for all OD pairs. All these figures were obtained using no trunk reservation for either traffic class.

Figures 6.6, 6.7, 6.8 and 6.9 show the result comparison of the simulation and numerical evaluation for the same 4-node network in table 6.1 but with trunk reservation of two units for each traffic class on all links. It can be seen that increasing trunk reservation makes more accurate the approximation decreasing the error for traffic class 1 below ten percent for all OD pairs and for traffic class 2 below 5%.

The performance evaluation of a 4-node fully connected network with two classes of traffic is shown in figures 6.10, 6.11, 6.12 and 6.13. The capacities on the links are $C_{01} = 10$, $C_{02} = 12$, $C_{03} = 8$, $C_{12} = 7$, $C_{13} = 14$, $C_{23} = 9$. The average capacity per link is of 10 units. In figures 6.10 and 6.12, the OD pair [1, 2] blocking probability for class 1 and class 2, respectively, is shown. The trunk reservation was 2 for each traffic class, i.e., $T = [2, 2]$. The traffic was fixed at 75 percent of the direct link capacity for all classes and all OD pairs but class 1 for OD pair [1, 2]. In figures 6.11 and 6.13, it is shown the error between the numerical evaluation given by the fixed point equations and the simulations for the same network and for traffic class 1 and

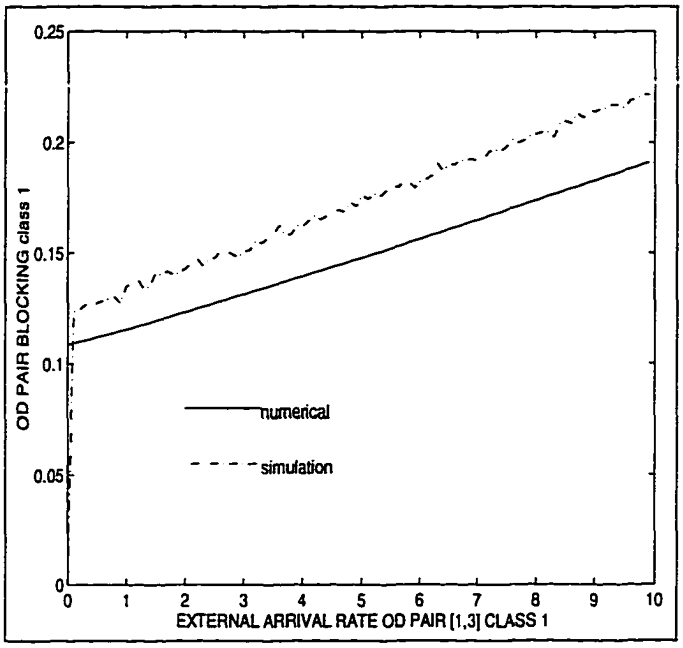

Figure 6.2: OD Pair [1, 3] Blocking Probability for Traffic Class 1, $T = [0, 0]$

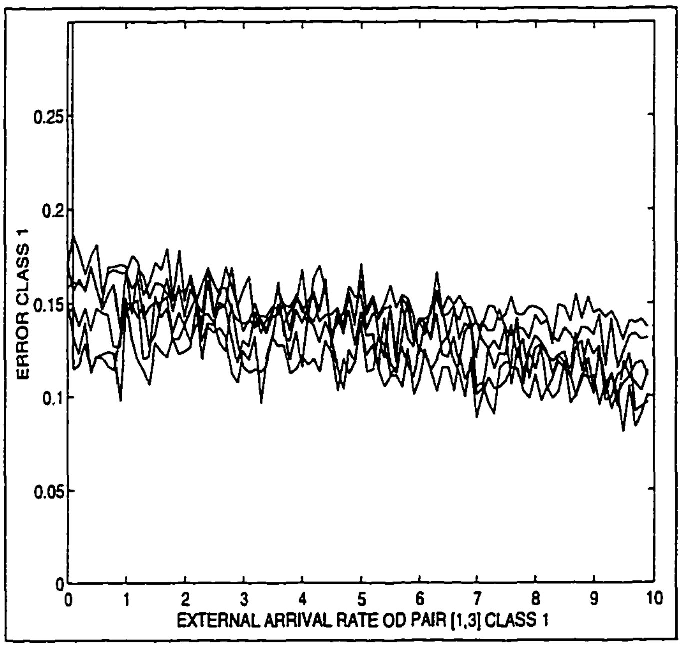

Figure 6.3: Performance Evaluation Accuracy for Traffic Class 1, $T = [0, 0]$

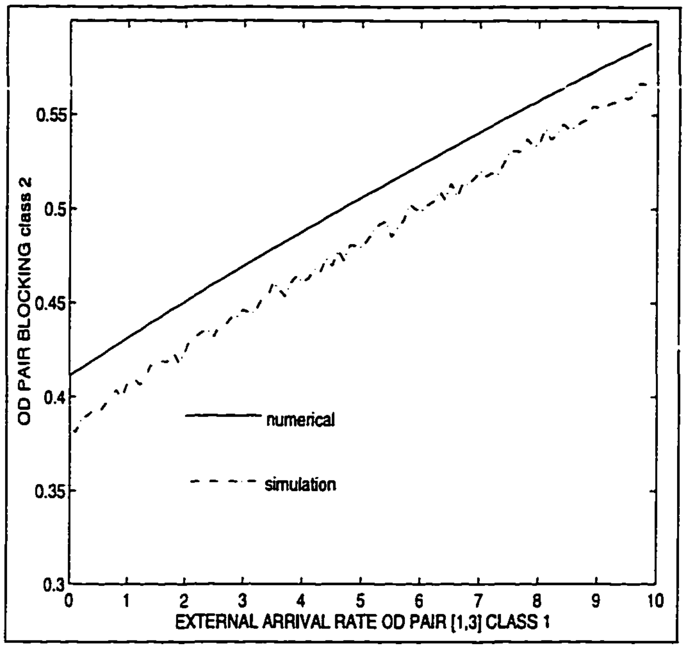

Figure 6.4: OD Pair [1, 3] Blocking Probability for Traffic Class 2, $T = [0, 0]$

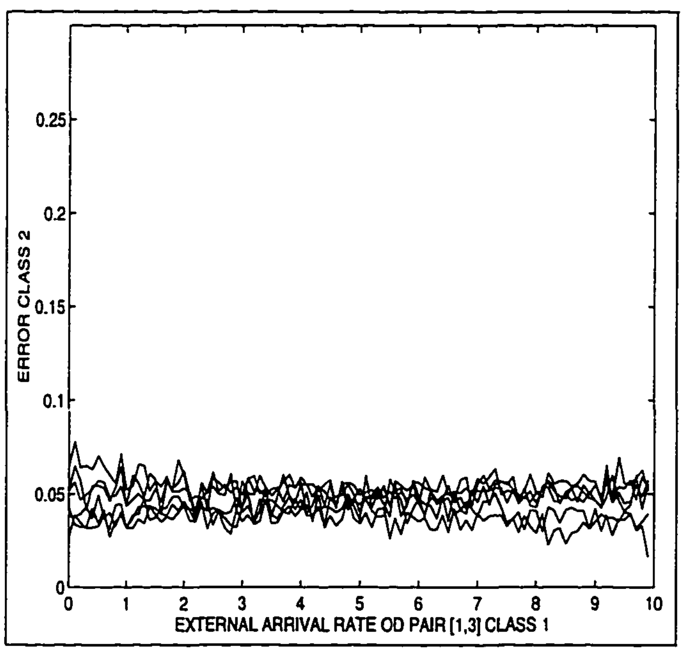

Figure 6.5: Performance Evaluation Accuracy for Traffic Class 2, $T = [0, 0]$

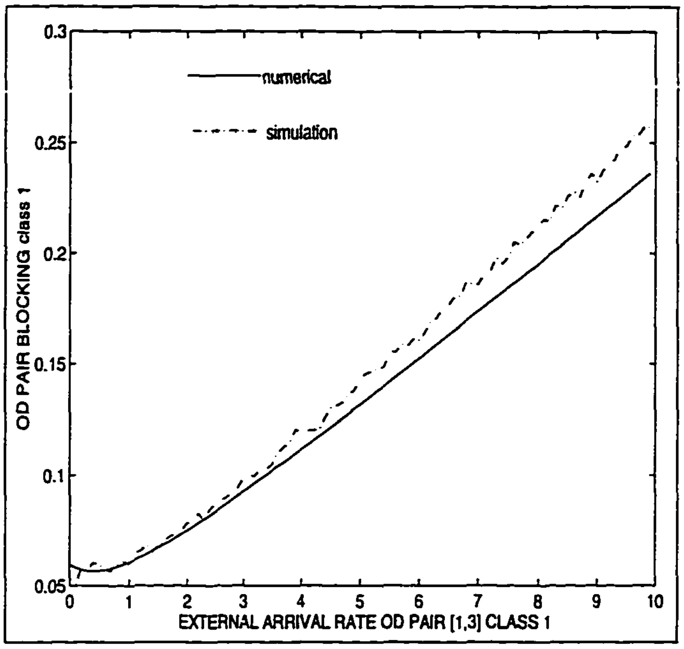

Figure 6.6: OD Pair [1, 3] Blocking Probability for Traffic Class 1, $T = [2, 2]$

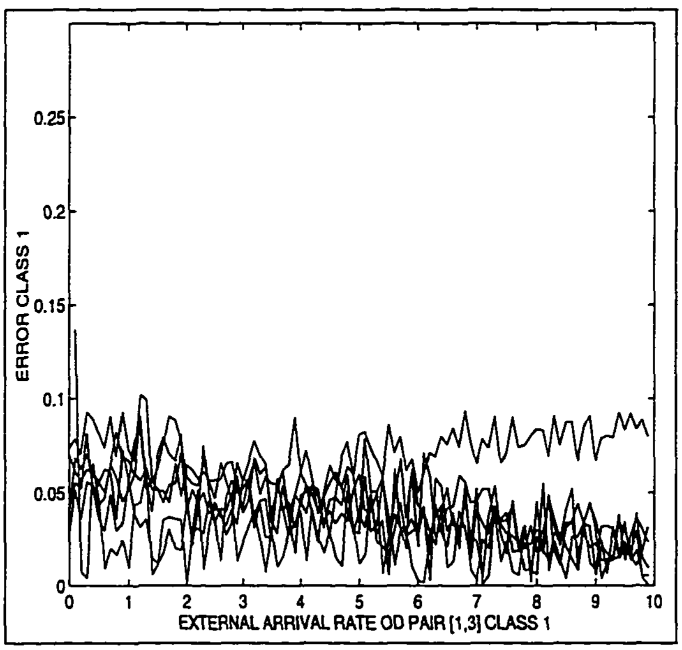

Figure 6.7: Performance Evaluation Accuracy for Traffic Class 1, $T = [2, 2]$

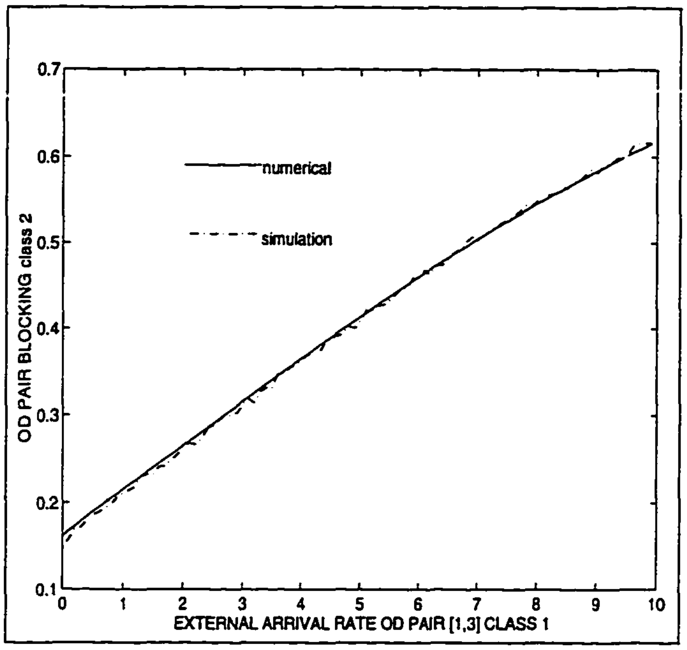

Figure 6.8: OD Pair [1, 3] Blocking Probability for Traffic Class 2, $T = [2, 2]$

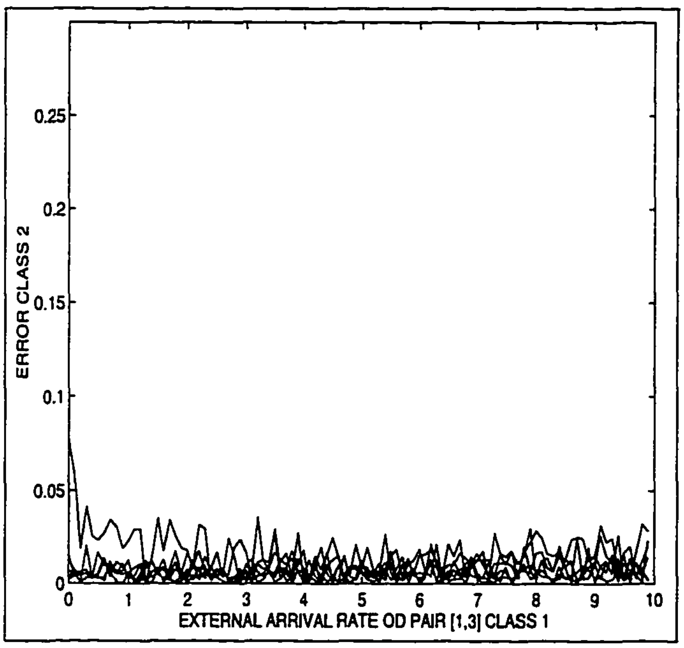

Figure 6.9: Performance Evaluation Accuracy for Traffic Class 2, $T = [2, 2]$

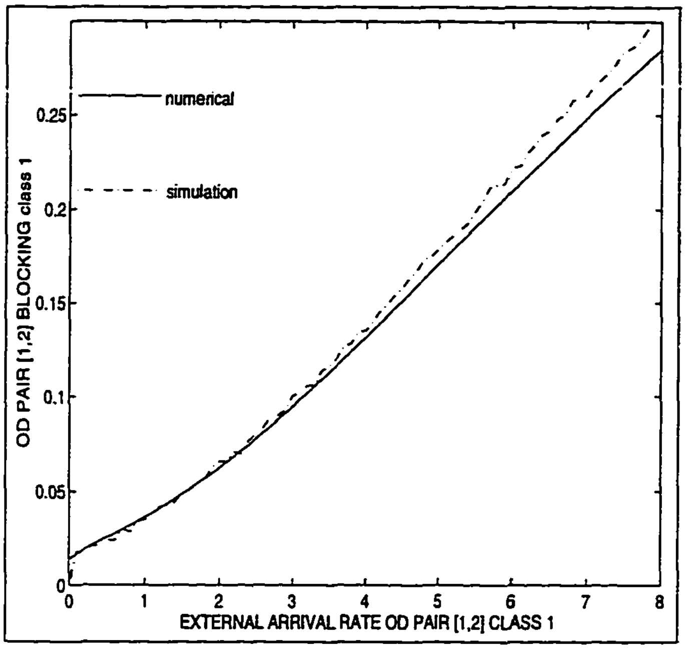

Figure 6.10: OD Pair [1, 2] Blocking Probability for Traffic Class 1

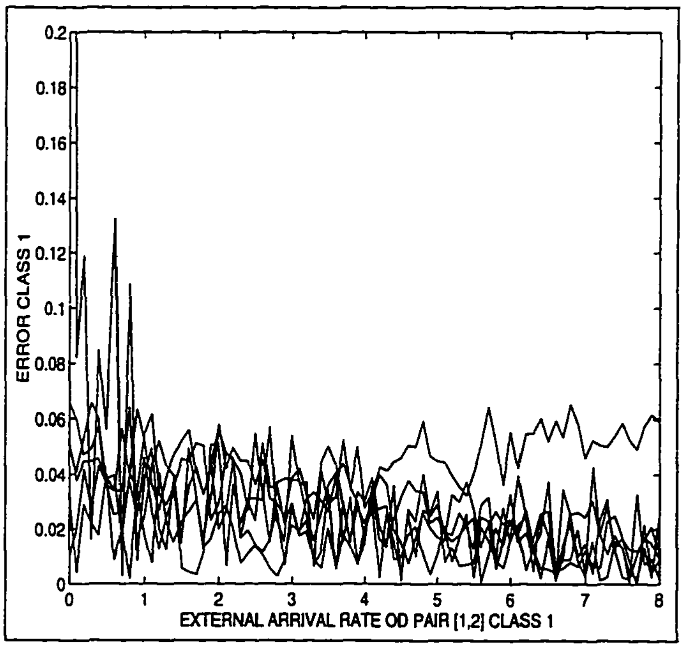

Figure 6.11: Performance Evaluation Accuracy for Traffic Class 1

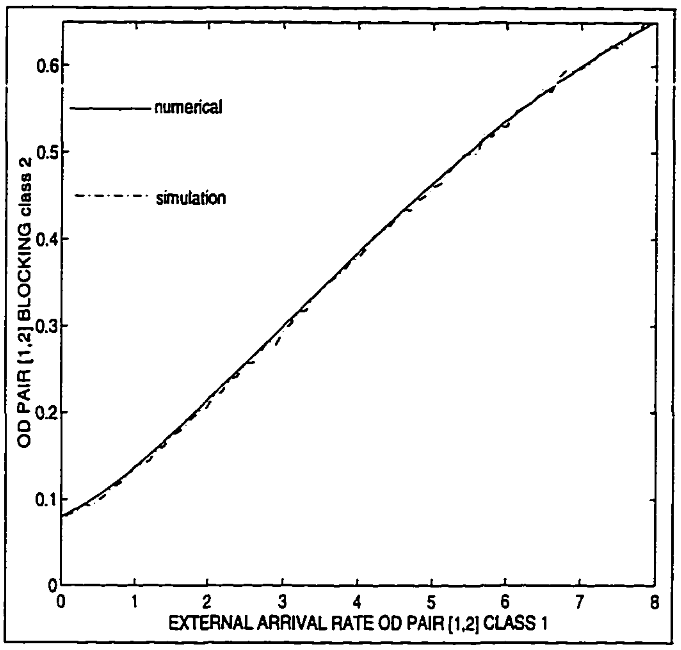

Figure 6.12: OD Pair [1, 2] Blocking Probability for Traffic Class 2

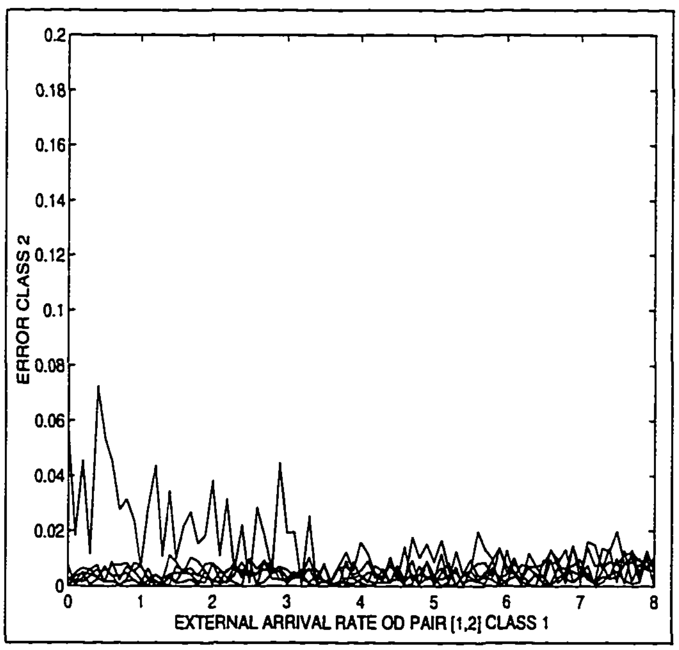

Figure 6.13: Performance Evaluation Accuracy for Traffic Class 2

2, respectively. It can be seen that the accuracy is within 10 percent for all OD pairs in the network and each traffic class. Figures 6.14 and 6.15 show the comparison of numerical results and simulation for the OD pair blocking probability of class 1 and class 2 traffic, respectively, for the same four-node network but with different service rates and different load, at 25the remaining OD pairs.

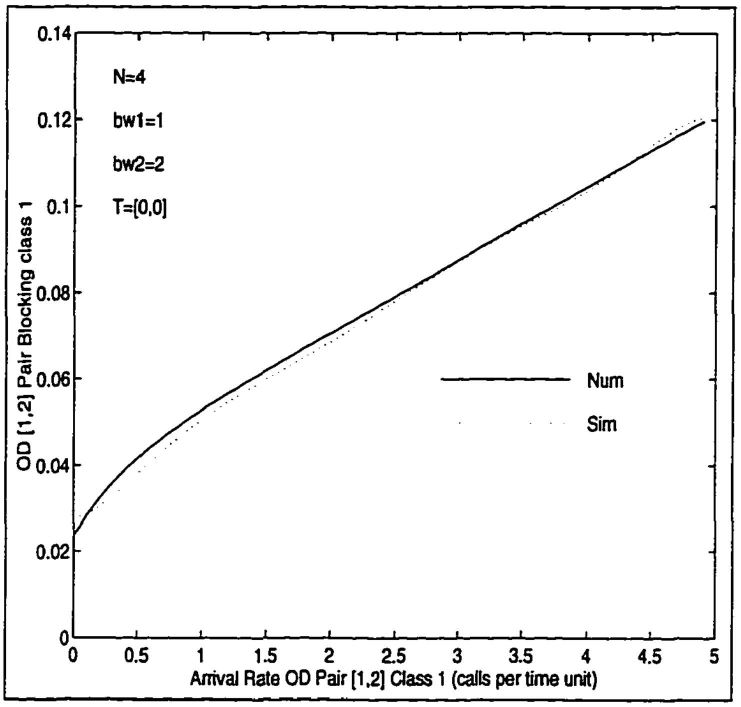

Figure 6.14: Class 1 Blocking Probability for the Four-Node Network. $\mu = [1, 1/5]$

Figures 6.16 and 6.17 show the comparison of numerical results while varying the service rate for the OD pair blocking probability of class 1 and class 2 traffic, respectively, for the same four-node network at a load of 25the remaining OD pairs.

6.3 Shadow Prices for Multi-rate LLR

In Chapter 4, shadow prices for single rate networks using LLR [8], ALBA [54], RALBA and MLLR [71] were presented. In that case, the shadow price was chosen to be with respect to the external arrival rates. In the present chapter we have

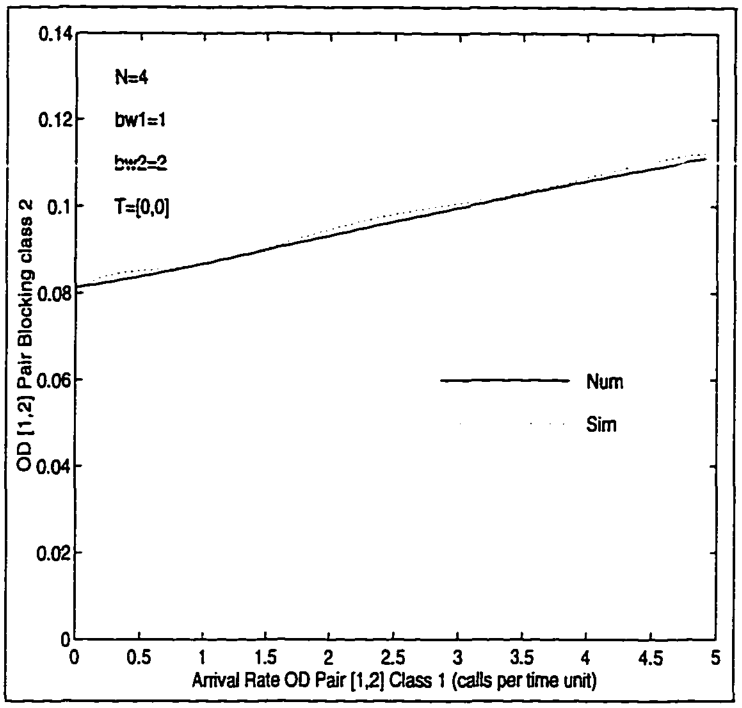

Figure 6.15: Class 2 Blocking Probability for the Four-Node Network. $\mu = [1, 1/5]$

different arrival rates for each of the classes of traffic in the network, shadow prices will be computed with respect to each of these rates for networks using LLR. In the following section the formulation of the shadow prices is presented.

6.3.1 Formulation

In this section, the formulation of the shadow prices for multirate LLR with K classes of traffic is presented. We consider the definition of the network rate of return, which is a function of the external arrival rates for all OD pairs and all classes of traffic and the OD pair blocking probabilities for each traffic class.

The Network rate of return is defined as follows

$$W(\underline{\lambda}, \underline{B}) = \sum_{[i,j] \in \mathcal{O}} \sum_{k=1}^{K} w_{ij}^{(k)} \lambda_{ij}^{(k)} \left(1 - B_{ij}^{(k)}\right), \tag{6.17}$$

where $w_{ij}^{(k)}$ is the revenue generated by accepting a call of type k of OD pair $[i, j]$ multiplied by the holding time of class k calls. The shadow price of this function

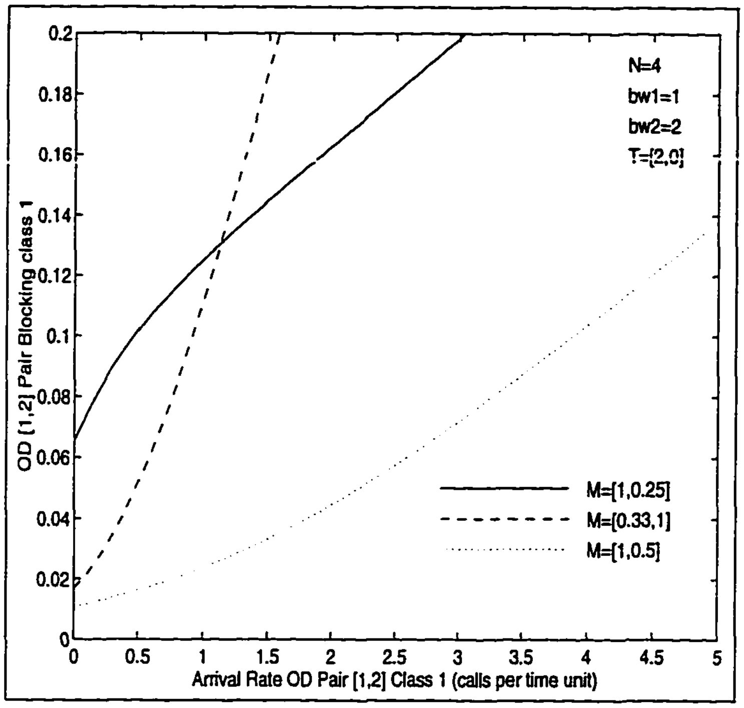

Figure 6.16: Class 1 Blocking Probability for the Four-Node Network.

with respect to the external arrival rate $\lambda_{ij}^{(k)}$ for OD pair $[i,j]$ and class k, is

$$\frac{dW(\underline{\lambda},\underline{\mathcal{B}})}{d\lambda_{ij}^{(k)}} = \frac{\partial W(\underline{\lambda},\underline{\mathcal{B}})}{\partial\lambda_{ij}^{(k)}} + \sum_{[r,s]\in\mathcal{O}} \sum_{v=1}^{K} \left[\frac{\partial W(\underline{\lambda},\underline{\mathcal{B}})}{\partial B_{rs}^{(v)}} \frac{dB_{rs}^{(v)}(\underline{p})}{d\lambda_{ij}^{(k)}}\right]. \tag{6.18}$$

The first term on the right hand side is obtained from equation (6.17) as

$$\frac{\partial W(\underline{\lambda},\underline{\mathcal{B}})}{\partial\lambda_{ij}^{(k)}} = w_{ij}^{(k)}\left(1 - B_{ij}^{(k)}(\underline{p})\right). \tag{6.19}$$

The second term of (6.18) contains two summations, one over the OD pairs and one over the traffic classes, both involve a partial derivative of the network rate of return with respect to the OD pair blocking probabilities, which can be obtained as

$$\frac{\partial W(\underline{\lambda},\underline{\mathcal{B}})}{\partial B_{rs}^{(v)}} = -\lambda_{rs}^{(v)} w_{rs}^{(v)}, \tag{6.20}$$

where $\underline{p}$ is the concatenation of all the probability distributions for all the states for each link $(a,b) \in \mathcal{L}$, i.e., $p_{ab}(\underline{\alpha}_{ab}, \mathbf{n})$, whose elements are the same as $p_{ab}(\mathbf{n})$

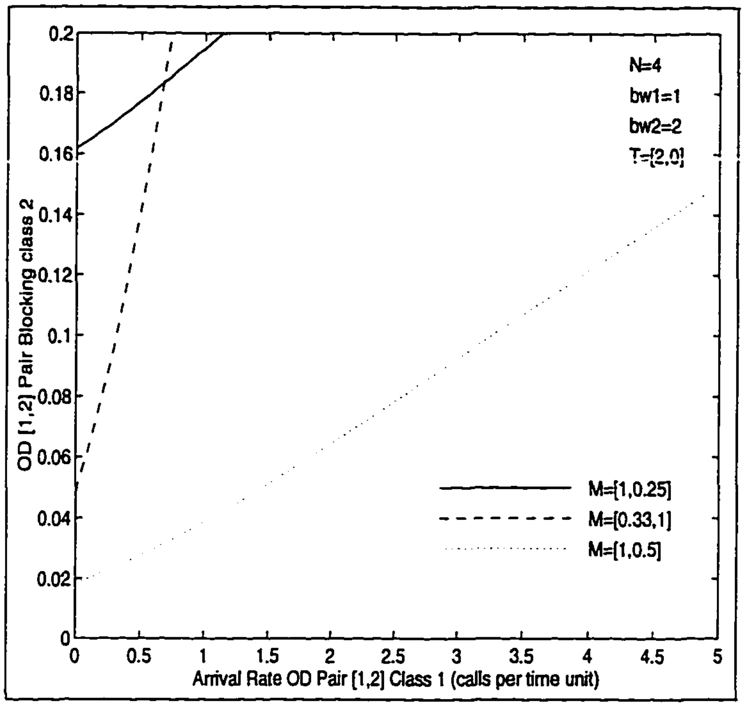

Figure 6.17: Class 2 Blocking Probability for the Four-Node Network.

obtained by the solution of the global balance equations in (6.5) and $\underline{\alpha}_{ab}$ is the vector that contains all the elements of $\alpha_{ab}^{(v)}(\mathrm{n})$ for all classes of traffic v and all states $\mathrm{n} \in \Omega_{ab}$. In equation (6.18) the total derivative of the OD pair blocking probabilities with respect to the external arrival rate and traffic class $\lambda_{ij}^{(k)}$ is needed, this can be calculated from

$$\frac{dB_{rs}^{(v)}(\underline{p})}{d\lambda_{ij}^{(k)}} = \sum_{(a,b)\in\mathcal{S}_{rs}} \sum_{\mathrm{n}\in\Omega_{ab}} \frac{\partial B_{rs}^{(v)}(\underline{p})}{\partial p_{ab}(\mathrm{n})} \cdot \frac{dp_{ab}(\underline{\alpha}_{ab}, \mathrm{n})}{d\lambda_{ij}^{(k)}}, \tag{6.21}$$

where $\mathcal{S}_{rs}$ is the set of links adjacent to link (r, s).

The last equation involves the partial derivative of the OD pair blocking probability with respect to the distribution for the states of the links this is obtained with the following expressions

$$\frac{\partial B_{rs}^{(v)}(\underline{p})}{\partial p_{rs}(\mathrm{n})} = \frac{B_{rs}^{(v)}(\underline{p})}{q_{rs}^{(v)}}, \quad \mathrm{n} \in \mathcal{B}_{rs}^{(v)}, \tag{6.22}$$

and for $d \in \mathcal{T}_{rs}$ and $\mathbf{n} \in \mathcal{U}_{rd}^{(v)}$, i.e., $\mathbf{n}$ is an unreserved state on link (r, d), the partial derivative can be obtained as

$$\frac{\partial B_{rs}^{(v)}(\underline{p})}{\partial p_{rd}(\mathbf{n})} = -\frac{B_{rs}^{(v)}(\underline{p}) \sum_{\mathbf{x} \in \mathcal{U}_{ds}^{(v)}} p_{ds}(\underline{\alpha}_{ds}, \mathbf{x})}{1 - \Big(\sum_{\mathbf{u} \in \mathcal{U}_{rd}^{(v)}} p_{rd}(\underline{\alpha}_{rd}, \mathbf{u})\Big)\Big(\sum_{\mathbf{x} \in \mathcal{U}_{ds}^{(v)}} p_{ds}(\underline{\alpha}_{ds}, \mathbf{x})\Big)}, \tag{6.23}$$

and with respect to $p_{ds}(\mathbf{n})$ where $\mathbf{n} \in \mathcal{U}_{ds}^{(v)}$, we have

$$\frac{\partial B_{rs}^{(v)}(\underline{p})}{\partial p_{ds}(\mathbf{n})} = -\frac{B_{rs}^{(v)}(\underline{p}) \sum_{\mathbf{x} \in \mathcal{U}_{rd}^{(v)}} p_{rd}(\underline{\alpha}_{rd}, \mathbf{x})}{1 - \Big(\sum_{\mathbf{u} \in \mathcal{U}_{rd}^{(v)}} p_{rd}(\underline{\alpha}_{rd}, \mathbf{u})\Big)\Big(\sum_{\mathbf{x} \in \mathcal{U}_{ds}^{(v)}} p_{ds}(\underline{\alpha}_{ds}, \mathbf{x})\Big)}. \tag{6.24}$$

The total derivative of the stationary distribution for the states of the links is calculated with the following expression

$$\frac{dp_{ab}(\underline{\alpha}_{ab}, \mathbf{n})}{d\lambda_{ij}^{(k)}} = \sum_{s=1}^{K} \sum_{\mathbf{m} \in \Omega_{ab}} \frac{\partial p_{ab}(\underline{\alpha}_{ab}, \mathbf{n})}{\partial \alpha_{ab}^{(s)}(\mathbf{m})} \frac{d\alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{d\lambda_{ij}^{(k)}}, \tag{6.25}$$

where it is shown the dependance of $\alpha_{ab}^{(s)}$ on the external arrival rates for all traffic classes and all OD pairs $\underline{\lambda}$, and on the stationary distribution for the states of all the links given by $\underline{p}$.

The first term in (6.25) on the right hand side is the partial derivative of the stationary distribution with respect to the offered traffic to the links. This can be obtained using the set of global balanced equations obtained from the K-dimensional Markov chain which are given in (6.5). In order to obtain this partial derivative, we have to get (6.5) for link (a, b) in the form

$$\begin{aligned} p_{ab}(\underline{\alpha}_{ab}, \mathbf{n}) \;=\; & \frac{1}{\sum_{i=1}^{K} n_i \mu_i + \sum_{i=1}^{K} f_{ab}^{(i)}(\mathbf{n}) \alpha_{ab}^{(i)}(\underline{\lambda}, \underline{p}, \mathbf{n})} \\ & \cdot \Big\{ \sum_{i=1}^{K} \mathcal{I}_{\{n_i > 0\}} \alpha_{ab}^{(i)}(\underline{\lambda}, \underline{p}, \mathbf{n}_i^-) p_{ab}(\underline{\alpha}_{ab}, \mathbf{n}_i^-) \\ & + \sum_{i=1}^{K} f_{ab}^{(i)}(\mathbf{n})(n_i + 1)\mu_i p_{ab}(\underline{\alpha}_{ab}, \mathbf{n}_i^+) \Big\}. \end{aligned} \tag{6.26}$$

Eliminate one of the equations and consider the normalizing equation. Obtain the partial derivatives of (6.26) as follows

$$\begin{aligned}
\frac{\partial p_{ab}(\underline{\alpha}_{ab},\mathbf{n})}{\partial \alpha_{ab}^{(s)}(\mathbf{m})} &= \frac{1}{\left[\sum_{i=1}^{K} n_i\mu_i + \sum_{i=1}^{K} f_{ab}^{(i)}(\mathbf{n})\alpha_{ab}^{(i)}(\underline{\lambda},\underline{p},\mathbf{n})\right]^2} \\
&\cdot \left\{\left[\sum_{i=1}^{K} n_i\mu_i + \sum_{i=1}^{K} f_{ab}^{(i)}(\mathbf{n})\alpha_{ab}^{(i)}(\underline{\lambda},\underline{p},\mathbf{n})\right]\right. \\
&\cdot \left[\sum_{i=1}^{K} \mathcal{I}_{\{n_i>0\}}\alpha_{ab}^{(i)}(\underline{\lambda},\underline{p},\mathbf{n}_i^-)\frac{\partial p_{ab}(\underline{\alpha}_{ab},\mathbf{n}_i^-)}{\partial \alpha_{ab}^{(s)}(\mathbf{m})}\right. \\
&+ \mathcal{I}_{\{\mathbf{m}=\mathbf{n}_s^-,\ \mathbf{n}_s>0\}}p_{ab}(\underline{\alpha}_{ab},\mathbf{n}_s^-) \\
&\left. + \sum_{i=1}^{K} f_{ab}^{(i)}(\mathbf{n})(n_i+1)\mu_i\frac{\partial p_{ab}(\underline{\alpha}_{ab},\mathbf{n}_i^+)}{\partial \alpha_{ab}^{(s)}(\mathbf{m})}\right] \\
&-\mathcal{I}_{\{\mathbf{m}=\mathbf{n}\}}\ f_{ab}^{(s)}(\mathbf{m})\left[\sum_{i=1}^{K} \mathcal{I}_{\{n_i>0\}}\alpha_{ab}^{(i)}(\underline{\lambda},\underline{p},\mathbf{n}_i^-)p_{ab}(\underline{\alpha}_{ab},\mathbf{n}_i^-)\right. \\
&\left.\left. +\sum_{i=1}^{K} f_{ab}^{(i)}(\mathbf{n})(n_i+1)\mu_i p_{ab}(\underline{\alpha}_{ab},\mathbf{n}_i^+)\right]\right\}, \qquad (6.27)
\end{aligned}$$

$$\sum_{\mathbf{n}\in\Omega_{ab}} \frac{\partial p_{ab}(\underline{\alpha}_{ab},\mathbf{n})}{\partial \alpha_{ab}^{(s)}(\mathbf{m})} = 0. \qquad (6.28)$$

Equation (6.27) results in the following

$$\begin{aligned}
\frac{\partial p_{ab}(\underline{\alpha}_{ab},\mathbf{n})}{\partial \alpha_{ab}^{(s)}(\mathbf{m})} &= \frac{1}{\sum_{i=1}^{K} n_i\mu_i + \sum_{i=1}^{K} f_{ab}^{(i)}(\mathbf{n})\alpha_{ab}^{(i)}(\underline{\lambda},\underline{p},\mathbf{n})} \\
&\cdot \left\{\sum_{i=1}^{K} \mathcal{I}_{\{n_i>0\}}\alpha_{ab}^{(i)}(\underline{\lambda},\underline{p},\mathbf{n}_i^-)\frac{\partial p_{ab}(\underline{\alpha}_{ab},\mathbf{n}_i^-)}{\partial \alpha_{ab}^{(s)}(\mathbf{m})}\right. \\
&+\sum_{i=1}^{K} f_{ab}^{(i)}(\mathbf{n})(n_i+1)\mu_i\frac{\partial p_{ab}(\underline{\alpha}_{ab},\mathbf{n}_i^+)}{\partial \alpha_{ab}^{(s)}(\mathbf{m})} \\
&+ \mathcal{I}_{\{\mathbf{m}=\mathbf{n}_s^-,\ \mathbf{n}_s>0\}}p_{ab}(\underline{\alpha}_{ab},\mathbf{n}_s^-) \\
&\left. -\mathcal{I}_{\{\mathbf{m}=\mathbf{n}\}}\ f_{ab}^{(s)}(\mathbf{m})p_{ab}(\underline{\alpha}_{ab},\mathbf{m})\right\}. \qquad (6.29)
\end{aligned}$$

This last equation, together with (6.28) results in a system of simultaneous linear equations on $\frac{\partial p_{ab}(\underline{\alpha}_{ab},\mathbf{n})}{\partial \alpha_{ab}^{(s)}(\mathbf{m})}$ for every $\mathbf{n} \in \Omega_{ab}$, $\mathbf{m} \notin \mathcal{B}_{ab}^{(s)}$, and we can solve it to find such partial derivatives. For the second term in (6.25), we compute the following expression for all state $\mathbf{m} \notin \mathcal{B}_{ab}^{(s)}$

$$\frac{d\alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{d\lambda_{ij}^{(k)}} = \frac{\partial\alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial\lambda_{ij}^{(k)}} + \sum_{(u,v)\in\mathcal{L}} \sum_{\mathbf{x}\in\Omega_{uv}} \frac{\partial\alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial p_{uv}(\mathbf{x})} \frac{dp_{uv}(\underline{\alpha}_{uv}, \mathbf{x})}{d\lambda_{ij}^{(k)}}. \quad (6.30)$$

The first partial derivative on the right hand side of (6.30) is given by

$$\frac{\partial\alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial\lambda_{ij}^{(k)}} = \begin{cases} 0, & \text{if } s \neq k, \ \forall\, \mathbf{m} \in \Omega_{ab}, \\ \mathcal{I}_{\{s=k\}}, & \text{if } (a,b) = (i,j), \ \forall\, \mathbf{m} \in \Omega_{ab}, \\ \mathcal{I}_{\{s=k\}} q_{ad} P_{ad}^{(s)}[(a,b), \mathbf{m}], & \text{if } (i,j) = (a,d), \ \forall\, d \in S_{ab}, \\ \mathcal{I}_{\{s=k\}} q_{db} P_{db}^{(s)}[(a,b), \mathbf{m}], & \text{if } (i,j) = (d,b), \ \forall\, \mathbf{m} \in \mathcal{U}_{ab}^{(s)}, \end{cases} \quad (6.31)$$

where $\mathcal{I}_{\{s=k\}} = 1$, if $s = k$ and 0 otherwise.

For the partial derivative of the offered traffic with respect to the distribution for the states of the links, we have several cases depending on the link with respect to which the partial derivative is obtained. For all these cases we will consider that link (a, b) is fixed and that its state is $\mathbf{m} \in \mathcal{U}_{ab}^{(s)}$ also fixed and we will get these derivatives for the offered traffic of class s to the link.

Using for all the different cases equations (6.13), (6.15) and (6.16), we get for links of the form (a, d) or (d, b), i.e., for the links adjacent to link (a, b), when they are in the blocked states

$$\frac{\partial\alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial p_{ad}(\mathbf{x})} = \lambda_{ad}^{(s)} P_{ad}^{(s)}[(a,b), \mathbf{m}], \quad \mathbf{x} \in \mathcal{B}_{ad}^{(s)}, \quad (6.32)$$

$$\frac{\partial\alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial p_{db}(\mathbf{x})} = \lambda_{db}^{(s)} P_{db}^{(s)}[(a,b), \mathbf{m}], \quad \mathbf{x} \in \mathcal{B}_{db}^{(s)}, \quad (6.33)$$

The second case considers the same links adjacent to link (a, b), but when their state is in the set of unreserved states, hence they are available to be used as part of a two-link alterante route for those OD pairs that can contribute to the offered traffic to link (a,b). Let the state of link (a,d) be $\mathbf{x} \in \overline{\mathcal{P}_{ad}^{ab}}(s,\mathbf{x},\mathbf{m})$ then from equations (6.13), (6.15) and (6.16), we get

$$\frac{\partial \alpha_{ab}^{(s)}(\underline{\lambda},\underline{p},\mathbf{m})}{\partial p_{ad}(\mathbf{x})} = \lambda_{db}^{(s)} q_{db}^{(s)} \cdot \prod_{w \in A_{db}^{-}(ab)} \left\{ 1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}_{ad}^{dw}}(s,\mathbf{r},\mathbf{x})} p_{dw}(\underline{\alpha}_{dw},\mathbf{r}) \right) \cdot \left(\sum_{\mathbf{z} \in \overline{\mathcal{M}_{ad}^{wb}}(s,\mathbf{z},\mathbf{x})} p_{wb}(\underline{\alpha}_{wb},\mathbf{z}) \right) \right\} \cdot \prod_{w \in A_{db}^{+}(ab)} \left\{ 1 - \left(\sum_{\mathbf{r} \in \mathcal{M}_{ad}^{dw}(s,\mathbf{r},\mathbf{x})} p_{dw}(\underline{\alpha}_{dw},\mathbf{r}) \right) \cdot \left(\sum_{\mathbf{z} \in \mathcal{M}_{ad}^{wb}(s,\mathbf{z},\mathbf{x})} p_{wb}(\underline{\alpha}_{wb},\mathbf{z}) \right) \right\}. \tag{6.34}$$

Now, let the state of link (a,d) be $\mathbf{x} \in \mathcal{M}_{ab}^{ad}(s,\mathbf{x},\mathbf{m})$ then we get

$$\frac{\partial \alpha_{ab}^{(s)}(\underline{\lambda},\underline{p},\mathbf{m})}{\partial p_{ad}(\mathbf{x})} = \lambda_{db}^{(s)} q_{db}^{(s)} \cdot \prod_{w \in A_{db}^{-}(ab)} \left\{ 1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}_{ab}^{dw}}(s,\mathbf{r},\mathbf{m})} p_{dw}(\underline{\alpha}_{dw},\mathbf{r}) \right) \cdot \left(\sum_{\mathbf{z} \in \overline{\mathcal{M}_{ab}^{wb}}(s,\mathbf{z},\mathbf{m})} p_{wb}(\underline{\alpha}_{wb},\mathbf{z}) \right) \right\} \cdot \prod_{w \in A_{db}^{+}(ab)} \left\{ 1 - \left(\sum_{\mathbf{r} \in \mathcal{M}_{ab}^{dw}(s,\mathbf{r},\mathbf{m})} p_{dw}(\underline{\alpha}_{dw},\mathbf{r}) \right) \cdot \left(\sum_{\mathbf{z} \in \mathcal{M}_{ab}^{wb}(s,\mathbf{z},\mathbf{m})} p_{wb}(\underline{\alpha}_{wb},\mathbf{z}) \right) \right\}, \tag{6.35}$$

similarly, the partial derivative with respect to the distribution of link (d,b) in state $\mathbf{x} \in \overline{\mathcal{P}_{db}^{ab}}(s,\mathbf{x},\mathbf{m})$ then from equations (6.13), (6.15) and (6.16), we get

$$\frac{\partial \alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial p_{db}(\mathbf{x})} = \lambda_{ad}^{(s)} q_{ad}^{(s)} \tag{6.36}$$

$$\cdot \prod_{w \in A_{ad}^{-}(ab)} \left\{ 1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}_{db}^{aw}}(s,\mathbf{r},\mathbf{x})} p_{aw}(\underline{\alpha}_{aw}, \mathbf{r}) \right) \right.$$

$$\left. \cdot \left(\sum_{\mathbf{z} \in \overline{\mathcal{M}_{db}^{wd}}(s,\mathbf{z},\mathbf{x})} p_{wd}(\underline{\alpha}_{wd}, \mathbf{z}) \right) \right\}$$

$$\cdot \prod_{w \in A_{ad}^{+}(ab)} \left\{ 1 - \left(\sum_{\mathbf{r} \in \mathcal{M}_{db}^{aw}(s,\mathbf{r},\mathbf{x})} p_{aw}(\underline{\alpha}_{aw}, \mathbf{r}) \right) \right.$$

$$\left. \cdot \left(\sum_{\mathbf{z} \in \mathcal{M}_{db}^{wd}(s,\mathbf{z},\mathbf{x})} p_{wd}(\underline{\alpha}_{wd}, \mathbf{z}) \right) \right\}.$$

Now, let the state of link (d, b) be $\mathbf{x} \in \mathcal{M}_{ab}^{bd}(s, \mathbf{x}, \mathbf{m})$ then we get

$$\frac{\partial \alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial p_{db}(\mathbf{x})} = \lambda_{ad}^{(s)} q_{ad}^{(s)} \tag{6.37}$$

$$\cdot \prod_{w \in A_{ad}^{-}(ab)} \left\{ 1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}_{ab}^{aw}}(s,\mathbf{r},\mathbf{m})} p_{aw}(\underline{\alpha}_{aw}, \mathbf{r}) \right) \right.$$

$$\left. \cdot \left(\sum_{\mathbf{z} \in \overline{\mathcal{M}_{ab}^{wd}}(s,\mathbf{z},\mathbf{m})} p_{wd}(\underline{\alpha}_{wd}, \mathbf{z}) \right) \right\}$$

$$\cdot \prod_{w \in A_{ad}^{+}(ab)} \left\{ 1 - \left(\sum_{\mathbf{r} \in \mathcal{M}_{ab}^{aw}(s,\mathbf{r},\mathbf{m})} p_{aw}(\underline{\alpha}_{aw}, \mathbf{r}) \right) \right.$$

$$\left. \cdot \left(\sum_{\mathbf{z} \in \mathcal{M}_{ab}^{wd}(s,\mathbf{z},\mathbf{m})} p_{wd}(\underline{\alpha}_{wd}, \mathbf{z}) \right) \right\},$$

The next case is when we consider link (a, b) fixed in state $\mathbf{m}$ as before, and the distribution of link (a, d) in state $\mathbf{x}$ but when it is part of a two-link alternate route of an OD pair of the form $[a, e]$, then we have to consider if the two-link alternate route (a, d), (d, e) precedes or succeeds the alternate route (a, b), (b, e) of OD pair $[a, e]$. In this case we get

$$
\begin{aligned}
\frac{\partial \alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial p_{ad}(\mathbf{x})} = & - \sum_{\substack{e:[a,e]\in\mathcal{O} \\ e\neq d}} \mathcal{I}_{\{(a,d)\in A_{ae}^{-}(ab)\}} \lambda_{ae}^{(s)} q_{ae}^{(s)} \qquad (6.38) \\
& \Bigg\{ \sum_{\mathbf{z}\,\in\,\overline{\mathcal{P}_{be}^{ab}}(s,\mathbf{z},\mathbf{m})} p_{be}(\mathbf{z})\, \mathcal{I}_{\{\mathbf{x}\in\overline{\mathcal{M}_{be}^{ad}}(s,\mathbf{x},\mathbf{z})\}} \cdot \left(\sum_{\mathbf{y}\in\overline{\mathcal{M}_{be}^{de}}(s,\mathbf{y},\mathbf{z})} p_{de}(\mathbf{y}) \right) \\
& \cdot \prod_{\substack{w\in A_{ae}^{-}(ab) \\ w\neq d}} \left[1 - \left(\sum_{\mathbf{r}\,\in\,\overline{\mathcal{M}_{be}^{aw}}(s,\mathbf{r},\mathbf{z})} p_{aw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t}\,\in\,\overline{\mathcal{M}_{be}^{we}}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right] \\
& \cdot \prod_{w\in A_{ae}^{+}(ab)} \left[1 - \left(\sum_{\mathbf{r}\,\in\,\mathcal{M}_{be}^{aw}(s,\mathbf{r},\mathbf{z})} p_{aw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t}\,\in\,\mathcal{M}_{be}^{we}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right] \\
& + \left(\sum_{\mathbf{z}\,\in\,\mathcal{M}_{ab}^{be}(s,\mathbf{z},\mathbf{m})} p_{be}(\mathbf{z}) \right) \cdot \left(\sum_{\mathbf{y}\in\overline{\mathcal{M}_{ab}^{de}}(s,\mathbf{y},\mathbf{m})} p_{de}(\mathbf{y}) \right) \mathcal{I}_{\{\mathbf{x}\in\overline{\mathcal{M}_{ab}^{ad}}(s,\mathbf{x},\mathbf{m})\}} \\
& \cdot \prod_{\substack{w\in A_{ae}^{-}(ab) \\ w\neq d}} \left[1 - \left(\sum_{\mathbf{r}\,\in\,\overline{\mathcal{M}_{ab}^{aw}}(s,\mathbf{r},\mathbf{m})} p_{aw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t}\,\in\,\overline{\mathcal{M}_{ab}^{we}}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right] \\
& \cdot \prod_{w\in A_{ae}^{+}(ab)} \left[1 - \left(\sum_{\mathbf{r}\,\in\,\mathcal{M}_{ab}^{aw}(s,\mathbf{r},\mathbf{m})} p_{aw}(\mathbf{r}) \right) \right. \\
& \left. \cdot \left(\sum_{\mathbf{t}\,\in\,\mathcal{M}_{ab}^{we}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right] \Bigg\} \\
& - \sum_{\substack{e:[a,e]\in\mathcal{O} \\ e\neq d}} \mathcal{I}_{\{(a,d)\in A_{ae}^{+}(ab)\}} \lambda_{ae}^{(s)} q_{ae}^{(s)} \\
& \Bigg\{ \sum_{\mathbf{z}\,\in\,\overline{\mathcal{P}_{be}^{ab}}(s,\mathbf{z},\mathbf{m})} p_{be}(\mathbf{z})\, \mathcal{I}_{\{\mathbf{x}\in\mathcal{M}_{be}^{ad}(s,\mathbf{x},\mathbf{z})\}} \cdot \left(\sum_{\mathbf{y}\in\mathcal{M}_{be}^{de}(s,\mathbf{y},\mathbf{z})} p_{de}(\mathbf{y}) \right) \\
& \cdot \prod_{w\in A_{ae}^{-}(ab)} \left[1 - \left(\sum_{\mathbf{r}\,\in\,\overline{\mathcal{M}_{be}^{aw}}(s,\mathbf{r},\mathbf{z})} p_{aw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t}\,\in\,\overline{\mathcal{M}_{be}^{we}}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right] \\
& \cdot \prod_{\substack{w\in A_{ae}^{+}(ab) \\ w\neq d}} \left[1 - \left(\sum_{\mathbf{r}\,\in\,\mathcal{M}_{be}^{aw}(s,\mathbf{r},\mathbf{z})} p_{aw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t}\,\in\,\mathcal{M}_{be}^{we}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right] \\
& + \left(\sum_{\mathbf{z}\,\in\,\mathcal{M}_{ab}^{be}(s,\mathbf{z},\mathbf{m})} p_{be}(\mathbf{z}) \right) \cdot \left(\sum_{\mathbf{y}\in\mathcal{M}_{ab}^{de}(s,\mathbf{y},\mathbf{m})} p_{de}(\mathbf{y}) \right) \mathcal{I}_{\{\mathbf{x}\in\mathcal{M}_{ab}^{ad}(s,\mathbf{x},\mathbf{m})\}}
\end{aligned}
$$

$$\cdot \prod_{w \in A^{-}_{ae}(ab)} \left[1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}^{aw}_{ab}}(s,\mathbf{r},\mathbf{m})} p_{aw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \overline{\mathcal{M}^{we}_{ab}}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right]$$
$$\cdot \prod_{\substack{w \in A^{+}_{ae}(ab) \\ w \neq d}} \left[1 - \left(\sum_{\mathbf{r} \in \mathcal{M}^{aw}_{ab}(s,\mathbf{r},\mathbf{m})} p_{aw}(\mathbf{r}) \right) \right.$$
$$\left. \left. \cdot \left(\sum_{\mathbf{t} \in \mathcal{M}^{we}_{ab}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right] \right\}.$$

Similarly, for link (d, b) in state $\mathbf{x}$ belonging to a two-link alternate route of OD pairs of the form $[e, b]$ we have

$$\frac{\partial \alpha^{(s)}_{ab}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial p_{db}(\mathbf{x})} = - \sum_{\substack{e:[e,b]\in\mathcal{O} \\ e \neq d}} \mathcal{I}_{\{(d,b)\in A^{-}_{eb}(ab)\}} \lambda^{(s)}_{eb} q^{(s)}_{eb} \tag{6.39}$$
$$\left\{ \sum_{\mathbf{z} \in \overline{\mathcal{P}^{ab}_{ae}}(s,\mathbf{z},\mathbf{m})} p_{ae}(\mathbf{z})\, \mathcal{I}_{\{\mathbf{x} \in \overline{\mathcal{M}^{db}_{ae}}(s,\mathbf{x},\mathbf{z})\}} \cdot \left(\sum_{\mathbf{y} \in \overline{\mathcal{M}^{de}_{ae}}(s,\mathbf{y},\mathbf{z})} p_{de}(\mathbf{y}) \right) \right.$$
$$\cdot \prod_{\substack{w \in A^{-}_{eb}(ab) \\ w \neq d}} \left[1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}^{bw}_{ae}}(s,\mathbf{r},\mathbf{z})} p_{bw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \overline{\mathcal{M}^{we}_{ae}}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right]$$
$$\cdot \prod_{w \in A^{+}_{eb}(ab)} \left[1 - \left(\sum_{\mathbf{r} \in \mathcal{M}^{bw}_{ae}(s,\mathbf{r},\mathbf{z})} p_{bw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \mathcal{M}^{we}_{ae}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right]$$
$$+ \left(\sum_{\mathbf{z} \in \mathcal{M}^{ae}_{ab}(s,\mathbf{z},\mathbf{m})} p_{ae}(\mathbf{z}) \right) \cdot \left(\sum_{\mathbf{y} \in \overline{\mathcal{M}^{de}_{ab}}(s,\mathbf{y},\mathbf{m})} p_{de}(\mathbf{y}) \right) \mathcal{I}_{\{\mathbf{x} \in \overline{\mathcal{M}^{db}_{ab}}(s,\mathbf{x},\mathbf{m})\}}$$
$$\cdot \prod_{\substack{w \in A^{-}_{eb}(ab) \\ w \neq d}} \left[1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}^{bw}_{ab}}(s,\mathbf{r},\mathbf{m})} p_{bw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \overline{\mathcal{M}^{we}_{ab}}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right]$$
$$\cdot \prod_{w \in A^{+}_{eb}(ab)} \left[1 - \left(\sum_{\mathbf{r} \in \mathcal{M}^{bw}_{ab}(s,\mathbf{r},\mathbf{m})} p_{bw}(\mathbf{r}) \right) \right.$$
$$\left. \left. \cdot \left(\sum_{\mathbf{t} \in \mathcal{M}^{we}_{ab}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right] \right\}$$
$$- \sum_{\substack{e:[e,b]\in\mathcal{O} \\ e \neq d}} \mathcal{I}_{\{(d,b)\in A^{+}_{eb}(ab)\}} \lambda^{(s)}_{eb} q^{(s)}_{eb}$$
$$\left\{ \sum_{\mathbf{z} \in \overline{\mathcal{P}^{ab}_{ae}}(s,\mathbf{z},\mathbf{m})} p_{ae}(\mathbf{z})\, \mathcal{I}_{\{\mathbf{x} \in \mathcal{M}^{db}_{ae}(s,\mathbf{x},\mathbf{z})\}} \cdot \left(\sum_{\mathbf{y} \in \mathcal{M}^{de}_{ae}(s,\mathbf{y},\mathbf{z})} p_{de}(\mathbf{y}) \right) \right.$$

$$
\begin{aligned}
&\cdot \prod_{w\in A^{-}_{eb}(ab)} \left[1-\left(\sum_{r\,\in\,\overline{\mathcal{M}^{bw}_{ae}}(s,r,z)} p_{bw}(r)\right)\left(\sum_{t\,\in\,\overline{\mathcal{M}^{we}_{ae}}(s,t,z)} p_{we}(t)\right)\right] \\
&\cdot \prod_{\substack{w\in A^{+}_{eb}(ab)\\ w\neq d}} \left[1-\left(\sum_{r\,\in\,\mathcal{M}^{bw}_{ae}(s,r,z)} p_{bw}(r)\right)\left(\sum_{t\,\in\,\mathcal{M}^{we}_{ae}(s,t,z)} p_{we}(t)\right)\right] \\
&+ \left(\sum_{z\,\in\,\mathcal{M}^{ae}_{ab}(s,z,m)} p_{ae}(z)\right)\cdot\left(\sum_{y\in\mathcal{M}^{de}_{ab}(s,y,m)} p_{de}(y)\right)\mathcal{I}_{\{x\in\mathcal{M}^{db}_{ab}(s,x,m)\}} \\
&\cdot \prod_{w\in A^{-}_{eb}(ab)} \left[1-\left(\sum_{r\,\in\,\overline{\mathcal{M}^{bw}_{ab}}(s,r,m)} p_{bw}(r)\right)\left(\sum_{t\,\in\,\overline{\mathcal{M}^{we}_{ab}}(s,t,m)} p_{we}(t)\right)\right] \\
&\cdot \prod_{\substack{w\in A^{+}_{eb}(ab)\\ w\neq d}} \left[1-\left(\sum_{r\,\in\,\mathcal{M}^{bw}_{ab}(s,r,m)} p_{bw}(r)\right)\right. \\
&\left.\left.\cdot \left(\sum_{t\,\in\,\mathcal{M}^{we}_{ab}(s,t,m)} p_{we}(t)\right)\right]\right\}.
\end{aligned}
$$

The last case is when link (a,b) in state $\mathbf{m}$ is fixed as before, and we consider the distribution of links which are not adjacent to link (a,b), but that belong to a two-link alternate route of an OD pair of the form $[a,e]$.

$$
\begin{aligned}
\frac{\partial\alpha^{(s)}_{ab}(\underline{\lambda},\underline{p},\mathbf{m})}{\partial p_{de}(\mathbf{x})} &= -\sum_{e:[a,e]\in\mathcal{O}} \mathcal{I}_{\{(d,e)\in A^{-}_{ae}(ab)\}}\lambda^{(s)}_{ae}q^{(s)}_{ae} \qquad (6.40) \\
&\left\{ \sum_{z\,\in\,\overline{\mathcal{P}^{ab}_{be}}(s,z,m)} p_{be}(z)\,\mathcal{I}_{\{x\in\overline{\mathcal{M}^{de}_{be}}(s,x,z)\}}\cdot\left(\sum_{y\in\overline{\mathcal{M}^{ad}_{be}}(s,y,z)} p_{ad}(y)\right)\right. \\
&\cdot \prod_{\substack{w\in A^{-}_{ae}(ab)\\ w\neq d}} \left[1-\left(\sum_{r\,\in\,\overline{\mathcal{M}^{aw}_{be}}(s,r,z)} p_{aw}(r)\right)\left(\sum_{t\,\in\,\overline{\mathcal{M}^{we}_{be}}(s,t,z)} p_{we}(t)\right)\right] \\
&\cdot \prod_{w\in A^{+}_{ae}(ab)} \left[1-\left(\sum_{r\,\in\,\mathcal{M}^{aw}_{be}(s,r,z)} p_{aw}(r)\right)\left(\sum_{t\,\in\,\mathcal{M}^{we}_{be}(s,t,z)} p_{we}(t)\right)\right] \\
&+ \left(\sum_{z\,\in\,\mathcal{M}^{be}_{ab}(s,z,m)} p_{be}(z)\right)\cdot\left(\sum_{y\in\overline{\mathcal{M}^{ad}_{ab}}(s,y,m)} p_{ad}(y)\right)\mathcal{I}_{\{x\in\overline{\mathcal{M}^{de}_{ab}}(s,x,m)\}} \\
&\cdot \prod_{\substack{w\in A^{-}_{ae}(ab)\\ w\neq d}} \left[1-\left(\sum_{r\,\in\,\overline{\mathcal{M}^{aw}_{ab}}(s,r,m)} p_{aw}(r)\right)\left(\sum_{t\,\in\,\overline{\mathcal{M}^{we}_{ab}}(s,t,m)} p_{we}(t)\right)\right]
\end{aligned}
$$

$$
\cdot \prod_{w \in A^{+}_{ae}(ab)} \left[1 - \left(\sum_{\mathbf{r} \in \mathcal{M}^{aw}_{ab}(s,\mathbf{r},\mathbf{m})} p_{aw}(\mathbf{r}) \right) \right.
$$

$$
\left. \cdot \left(\sum_{\mathbf{t} \in \mathcal{M}^{we}_{ab}(s,\mathbf{t},\mathbf{m})} \bar{p}_{we}(\mathbf{t}) \right) \right] \Bigg\}
$$

$$
- \sum_{e:[a,e] \in \mathcal{O}} \mathcal{I}_{\{(d,e) \in A^{+}_{ae}(ab)\}} \lambda^{(s)}_{ae} q^{(s)}_{ae}
$$

$$
\Bigg\{ \sum_{\mathbf{z} \in \overline{\mathcal{P}^{ab}_{be}}(s,\mathbf{z},\mathbf{m})} p_{be}(\mathbf{z}) \, \mathcal{I}_{\{\mathbf{x} \in \mathcal{M}^{de}_{be}(s,\mathbf{x},\mathbf{z})\}} \cdot \left(\sum_{\mathbf{y} \in \mathcal{M}^{ad}_{be}(s,\mathbf{y},\mathbf{z})} p_{ad}(\mathbf{y}) \right)
$$

$$
\cdot \prod_{w \in A^{-}_{ae}(ab)} \left[1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}^{aw}_{be}}(s,\mathbf{r},\mathbf{z})} p_{aw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \overline{\mathcal{M}^{we}_{be}}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right]
$$

$$
\cdot \prod_{\substack{w \in A^{+}_{ae}(ab) \\ w \neq d}} \left[1 - \left(\sum_{\mathbf{r} \in \mathcal{M}^{aw}_{be}(s,\mathbf{r},\mathbf{z})} p_{aw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \mathcal{M}^{we}_{be}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right]
$$

$$
+ \left(\sum_{\mathbf{z} \in \mathcal{M}^{be}_{ab}(s,\mathbf{z},\mathbf{m})} p_{be}(\mathbf{z}) \right) \cdot \left(\sum_{\mathbf{y} \in \mathcal{M}^{ad}_{ab}(s,\mathbf{y},\mathbf{m})} p_{ad}(\mathbf{y}) \right) \mathcal{I}_{\{\mathbf{x} \in \mathcal{M}^{de}_{ab}(s,\mathbf{x},\mathbf{m})\}}
$$

$$
\cdot \prod_{w \in A^{-}_{ae}(ab)} \left[1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}^{aw}_{ab}}(s,\mathbf{r},\mathbf{m})} p_{aw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \overline{\mathcal{M}^{we}_{ab}}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right]
$$

$$
\cdot \prod_{\substack{w \in A^{+}_{ae}(ab) \\ w \neq d}} \left[1 - \left(\sum_{\mathbf{r} \in \mathcal{M}^{aw}_{ab}(s,\mathbf{r},\mathbf{m})} p_{aw}(\mathbf{r}) \right) \right.
$$

$$
\left. \cdot \left(\sum_{\mathbf{t} \in \mathcal{M}^{we}_{ab}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right] \Bigg\}
$$

$$
- \sum_{e:[e,b] \in \mathcal{O}} \mathcal{I}_{\{(d,e) \in A^{-}_{eb}(ab)\}} \lambda^{(s)}_{eb} q^{(s)}_{eb}
$$

$$
\Bigg\{ \sum_{\mathbf{z} \in \overline{\mathcal{P}^{ab}_{ae}}(s,\mathbf{z},\mathbf{m})} p_{ae}(\mathbf{z}) \, \mathcal{I}_{\{\mathbf{x} \in \overline{\mathcal{M}^{de}_{ae}}(s,\mathbf{x},\mathbf{z})\}} \cdot \left(\sum_{\mathbf{y} \in \overline{\mathcal{M}^{bd}_{ae}}(s,\mathbf{y},\mathbf{z})} p_{bd}(\mathbf{y}) \right)
$$

$$
\cdot \prod_{\substack{w \in A^{-}_{eb}(ab) \\ w \neq d}} \left[1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}^{bw}_{ae}}(s,\mathbf{r},\mathbf{z})} p_{bw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \overline{\mathcal{M}^{we}_{ae}}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right]
$$

$$
\cdot \prod_{w \in A^{+}_{eb}(ab)} \left[1 - \left(\sum_{\mathbf{r} \in \mathcal{M}^{bw}_{ae}(s,\mathbf{r},\mathbf{z})} p_{bw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \mathcal{M}^{we}_{ae}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right]
$$

$$
\begin{aligned}
&+ \left(\sum_{\mathbf{z} \in \mathcal{M}_{ab}^{ae}(s,\mathbf{z},\mathbf{m})} p_{ae}(\mathbf{z}) \right) \cdot \left(\sum_{\mathbf{y} \in \overline{\mathcal{M}_{ab}^{bd}}(s,\mathbf{y},\mathbf{m})} p_{bd}(\mathbf{y}) \right) \mathcal{I}_{\{\mathbf{x} \in \overline{\mathcal{M}_{ab}^{de}}(s,\mathbf{x},\mathbf{m})\}} \\
&\cdot \prod_{\substack{w \in A_{eb}^{-}(ab) \\ w \neq d}} \left[1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}_{ab}^{bw}}(s,\mathbf{r},\mathbf{m})} p_{bw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \overline{\mathcal{M}_{ab}^{we}}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right] \\
&\cdot \prod_{w \in A_{eb}^{+}(ab)} \left[1 - \left(\sum_{\mathbf{r} \in \mathcal{M}_{ab}^{bw}(s,\mathbf{r},\mathbf{m})} p_{bw}(\mathbf{r}) \right) \right. \\
&\left. \cdot \left(\sum_{\mathbf{t} \in \mathcal{M}_{ab}^{we}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right] \Bigg\} \\
&- \sum_{e:[e,b] \in \mathcal{O}} \mathcal{I}_{\{(d,e) \in A_{eb}^{+}(ab)\}} \lambda_{eb}^{(s)} q_{eb}^{(s)} \\
&\Bigg\{ \sum_{\mathbf{z} \in \overline{\mathcal{P}_{ae}^{ab}}(s,\mathbf{z},\mathbf{m})} p_{ae}(\mathbf{z}) \, \mathcal{I}_{\{\mathbf{x} \in \mathcal{M}_{ae}^{de}(s,\mathbf{x},\mathbf{z})\}} \cdot \left(\sum_{\mathbf{y} \in \mathcal{M}_{be}^{bd}(s,\mathbf{y},\mathbf{z})} p_{bd}(\mathbf{y}) \right) \\
&\cdot \prod_{w \in A_{eb}^{-}(ab)} \left[1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}_{ae}^{bw}}(s,\mathbf{r},\mathbf{z})} p_{bw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \overline{\mathcal{M}_{ae}^{we}}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right] \\
&\cdot \prod_{\substack{w \in A_{eb}^{+}(ab) \\ w \neq d}} \left[1 - \left(\sum_{\mathbf{r} \in \mathcal{M}_{ae}^{bw}(s,\mathbf{r},\mathbf{z})} p_{bw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \mathcal{M}_{ae}^{we}(s,\mathbf{t},\mathbf{z})} p_{we}(\mathbf{t}) \right) \right] \\
&+ \left(\sum_{\mathbf{z} \in \mathcal{M}_{ab}^{ae}(s,\mathbf{z},\mathbf{m})} p_{ae}(\mathbf{z}) \right) \cdot \left(\sum_{\mathbf{y} \in \mathcal{M}_{ab}^{bd}(s,\mathbf{y},\mathbf{m})} p_{bd}(\mathbf{y}) \right) \mathcal{I}_{\{\mathbf{x} \in \mathcal{M}_{ab}^{de}(s,\mathbf{x},\mathbf{m})\}} \\
&\cdot \prod_{w \in A_{eb}^{-}(ab)} \left[1 - \left(\sum_{\mathbf{r} \in \overline{\mathcal{M}_{ab}^{bw}}(s,\mathbf{r},\mathbf{m})} p_{bw}(\mathbf{r}) \right) \left(\sum_{\mathbf{t} \in \overline{\mathcal{M}_{ab}^{we}}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right] \\
&\cdot \prod_{\substack{w \in A_{eb}^{+}(ab) \\ w \neq d}} \left[1 - \left(\sum_{\mathbf{r} \in \mathcal{M}_{ab}^{bw}(s,\mathbf{r},\mathbf{m})} p_{bw}(\mathbf{r}) \right) \right. \\
&\left. \cdot \left(\sum_{\mathbf{t} \in \mathcal{M}_{ab}^{we}(s,\mathbf{t},\mathbf{m})} p_{we}(\mathbf{t}) \right) \right] \Bigg\}.
\end{aligned}
$$

Substituting equations (6.31) - (6.41) into (6.30) and taking this result to (6.25), we find the set of linear simultaneous equations as in the case of single-rate LLR. The system is given by

$$\frac{dp_{ab}(\underline{\alpha}_{ab}, \mathbf{n})}{d\lambda_{ij}^{(k)}} = \sum_{s=1}^{K} \sum_{\mathbf{m}\in\Omega_{ab}} \frac{\partial p_{ab}(\underline{\alpha}_{ab}, \mathbf{n})}{\partial \alpha_{ab}^{(s)}(\mathbf{m})} \frac{\partial \alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial \lambda_{ij}^{(k)}} + \sum_{s=1}^{K} \sum_{\mathbf{m}\in\Omega_{ab}} \frac{\partial p_{ab}(\underline{\alpha}_{ab}, \mathbf{n})}{\partial \alpha_{ab}^{(s)}(\mathbf{m})} \sum_{(u,v)\in\mathcal{L}} \sum_{\mathbf{x}\in\Omega_{uv}} \frac{\partial \alpha_{ab}^{(s)}(\underline{\lambda}, \underline{p}, \mathbf{m})}{\partial p_{uv}(\mathbf{x})} \frac{dp_{uv}(\underline{\alpha}_{uv}, \mathbf{x})}{d\lambda_{ij}^{(k)}}. \quad (6.41)$$

Solving for $\frac{dp_{ab}(\underline{\alpha}_{ab}, \mathbf{n})}{d\lambda_{ij}^{(k)}}$ in (6.42) and taking the result together with equations (6.19) - (6.24) into (6.18) will compute the shadow prices of the network rate of return $W(\underline{\lambda}, \underline{\mathcal{B}})$. Figures 6.18 and 6.19 show the rate of return and shadow price with respect to the external arrival rate of OD pair [1, 2] for the four-node network of the examples in the previous section. It can be seen that they are close. To show the agreement between simulation and analysis, confidence intervals for the shadow price are included. The confidence intervals used were of 90%.

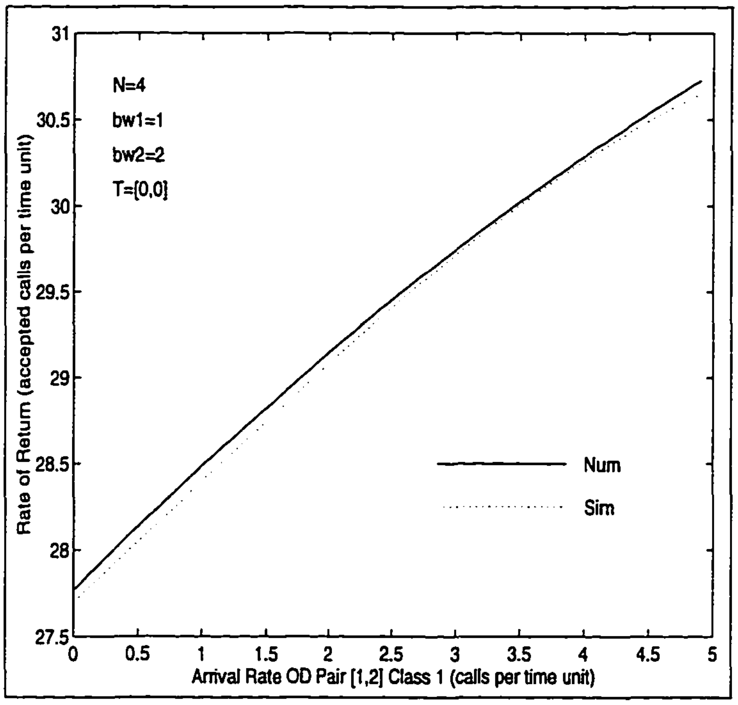

Figure 6.18: Rate of Return for the Four-Node Network. $\mu = [1, 1/5]$

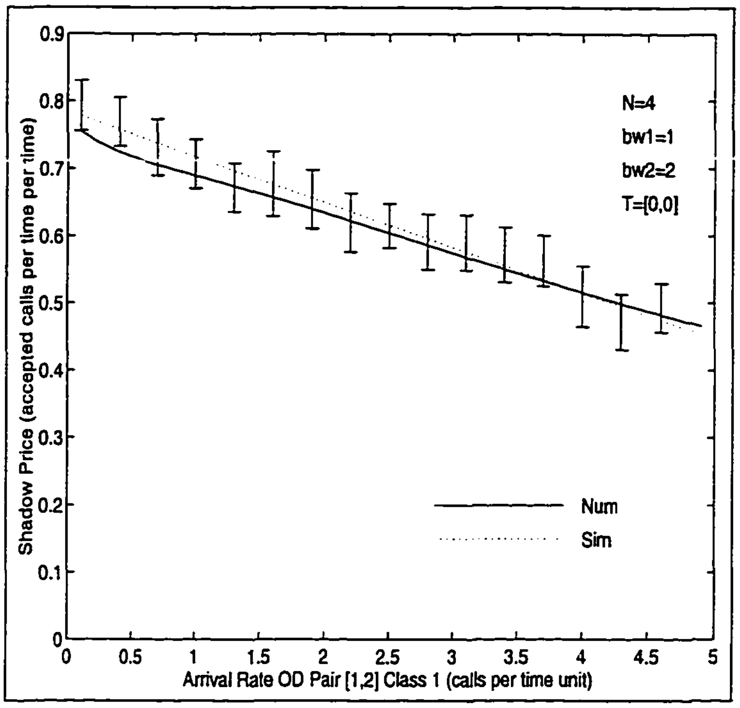

Figure 6.19: Shadow Price for the Four-Node Network. $\mu = [1, 1/5]$

6.4 Sum Capacity for Multirate LLR

The sum capacity for multirate LLR is not much different from the case of single rate LLR. The sum capacity is calculated for both traffic classes at the same time, and gives as a result the largest sum of external arrival rates for both classes such that the OD pair blocking probability is maintained at a prescribed level maximizing the network rate of return. This is obtained by solving the constrained nonlinear optimization problem defined as follows

$$\begin{aligned} \max_{\underline{\lambda}} \quad W(\underline{\lambda}, \underline{\mathcal{B}}) &= \sum_{[i,j]\in\mathcal{O}} \sum_{k=1}^{K} w_{ij}^{(k)} \lambda_{ij}^{(k)} \left(1 - B_{ij}^{(k)}\right) \\ \text{subject to} \quad & \underline{\mathcal{B}} \leq \underline{\eta}, \\ & \underline{\lambda} \geq \underline{0}. \end{aligned} \tag{6.42}$$

where $\mathcal{B}$ is the vector whose elements are the OD pair blocking probabilities for both traffic classes, and $\underline{\eta}$ is the prescribed level of blocking desired. The solution for the above optimization problem gives the maximum traffic that the network can carry for a given blocking probability vector. The optimization is achieved by using the shadow prices in a gradient descent algorithm that gives the direction in which the vector of exogenous arrival rates has to be varied to get the desired maximization. The specific algorithm used was the variable metric method [62] using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update formula. The step size is obtained by doing a line minimization of Powell's penalty function [62] and the algorithm is stopped when the improvement in the objective function $W(\underline{\lambda}, \underline{\mathcal{B}})$ is less than 10^{-7}, each blocking probability, $B_i^{(m)}$ is in the interval, $(\eta_i - 10^{-4}, \eta_i)$, and the change in each of the decision variables $\lambda_j^{(m)}$, $m = 1, 2, \ldots, K$ is no greater than 10^{-4}.

The sum capacity for the four-node network of previous examples with $\mu = [1, 1/2]$ is shown in Figure 6.20 for different values of trunk reservation. It can be seen that as the reservation of the class with higher bandwidth increases, the sum capacity decreases, and as the reservation for the class with low bandwidth increases, the sum capacity increases.

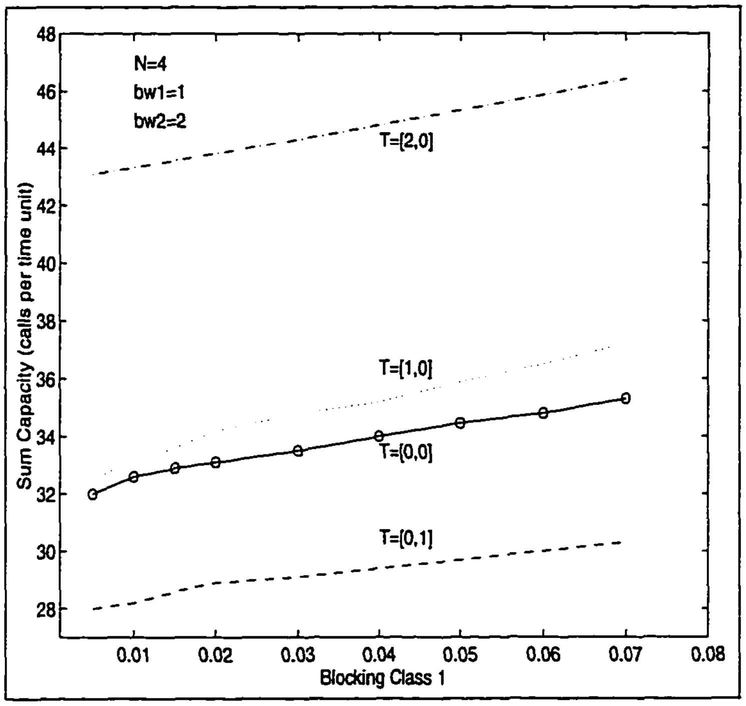

Figure 6.20: Sum Capacity for the Four-Node Network.

Chapter 7

Shadow Prices and Wireless Networks

In this chapter, the shadow price methodology is extended to be applied to wireless networks. The same techniques used in previous chapters help find the sum revenue of a network with single and multiple classes of customers. Finally, two of the most important applications are analyzed, the optimal number of channels to be assigned in order to keep the quality of service at good levels and the optimal channel reservation levels to give priority to calls already in progress in the network.

First, we present the fixed point model for wireless networks with multiple classes of customers.

7.1 Introduction

Wireless networks are getting deployed at an increasing rate and may soon rival wireline networks in their coverage. The growth of wireless networks reflects a paradigmatic shift of communication from point-to-point to person-to-person. While wireless networks provide much convenience for customers in terms of accomodating mobility, their deployment has generated new problems peculiar to wireless networks. For instance, bandwidth in a wireless network is an extremely scarce resource and much effort is expended towards its efficient use.

As such, the performance analysis of wireless networks, while using techniques similar to those used in the performance analysis of wireline networks, focuses on issues which are specific to wireless networks and mobility, [10].

One such mobility-specific issue in wireless networks is the treatment of handoffs where, because of a user's mobility, the call has to be handed off from one base station to another. If the new base station does not have a channel available, the handoff call will be blocked. Typically, on account of customer indignation, the rejection of a handoff is considered to be more detrimental than the rejection of new incoming calls. Thus, the handoff blocking probability is an important criterion for the performance of wireless networks.

Essentially three methods are used for the admission of handoffs and new call arrivals. One treats handoff calls and new calls equally for occupancy of the channels, the second reserves channels in each cell to give priority to handoffs and the third sends handoffs to a queue if no channel is available. Performance evaluation algorithms for these strategies have been introduced, for example in [28] for the reservation strategy and the queueing strategy and in [53] for the reservation and no reservation strategy. In this paper we use a model with reservation to give priority to handoff calls versus new calls and with Fixed Channel Allocation (FCA), i.e., every cell is assigned a fixed number of channels. For other kind of channel assignments as well as their analyses see [64], [78] and [79].

In general, the analysis techniques used for evaluating the performance of wireless networks require fixed point computations to obtain blocking probability and/or handoff drop probability. The use of fixed point computations and the consequent implicitness of the dependence of the blocking probability or the handoff drop performance on the entire network traffic obscure the effects of variables such as exogenous inputs on the performance measures. In this paper we use the concept of shadow price [35], [36] to evaluate these.

In [35], the author calculates shadow prices, i.e., the derivative of the network rate of return with respect to external traffic and with respect to link capacities, for circuit-switched networks. In [36], this work is extended to include the case of trunk reservation and in [67] and [70] to the case of adaptive routing. Shadow prices can be used in applications such as in algorithms to aid capacity expansion decisions [22], [59], in pricing policy [52] and for the apportionment of revenue between various sections of a network, [76], to mention some.

In order to calculate shadow prices, an extension of a fixed point algorithm similar to the one in [53] is used to evaluate performance given by the new call blocking and the handoff drop probabilities for multiple classes of customers with different service rates, different call arrival rates and different bandwidth requirements for each class. Also, the model considers differently the mobility of new customers to that of customers who have already handed off to an adjacent cell as is done in [48] for store-and-forward networks. The forced termination probability as introduced in [53] is used in the performance evaluation and its relationship with new call blocking probability is evaluated. We indicate the accuracy of the algorithm by comparing analytical results to simulations. We present numerical results for several examples and suggest how they can be utilized for network pricing. We use shadow prices with respect to new call arrival to calculate sum revenue for FCA with reservation for single and multi-rate wireless networks as the solution of a nonlinear constrained maximization problem where the solution is achieved using the shadow prices in a gradient descent algorithm.

In Section 7.2 we present the fixed point algorithm for the multirate FCA scheme with reservations. In Section 7.3 we develop the shadow price methodology for the fixed point model and present the complexity of the algorithm. In Section 7.4 we indicate the accuracy of the algorithm by comparing analytical results to simulations. We present numerical results for several examples and suggest how they can be utilized for network pricing. We use shadow prices with respect to new call arrival to calculate sum revenue for FCA with reservation for single and multi-rate wireless networks as the solution of a nonlinear constrained maximization problem where the solution is achieved using the shadow prices in a gradient descent algorithm.

7.2 Model for Multirate Wireless Networks

Consider an asymmetric cellular network with fixed channel assignment where $\mathcal{N}$ is the set of cells and N, the total number of cells. Each cell i has C_i channels assigned to it. Let $\mathcal{A}_i$ be the set of cells adjacent to cell i. There are M classes of traffic which share the network resources. Let b_m be the number of channels required by traffic of class m, $m = 1, 2, \ldots, M$, $0 < b_1 \leq b_2 \leq \cdots \leq b_M$. The new call arrival process of class m to cell i is a Poisson Process with mean $\lambda_{i,m}$ independent of other new call arrival processes. The time a call of class m remains in cell i, the dwell time, is a random variable with exponential distribution and mean $1/\mu_{i,m}$ and it is independent of earlier arrival times, call durations and elapsed times of other users. At the end of a dwell time a call may attempt a handoff to an adjacent cell or leave the network. In this model a *new* call in progress in any cell is treated differently from a *handoff* call for the purposes of termination. We have two *levels* of probabilities for each cell and for each class of traffic m: let $q^{(1)}_{ij,m}$ be the probability that a *new* call in progress in cell i after completing its dwell time *goes* to cell j, i.e., there is a first handoff from cell i to cell j of class m and let $q^{(2)}_{ij,m}$ be the probability that a *handoff* call of class m in progress in cell i after completing its dwell time *goes* to cell j. If cell i and cell j are not adjacent then $q^{(s)}_{ij,m} = 0$ for $s = 1, 2$ and $m = 1, \ldots, M$. Let $q^{(1)}_{ii,m}$ be the probability of a class m departure from the network from cell i when the call in progress is a new call and $q^{(2)}_{ii,m}$ be the probability of a class m departure from the network from cell i when the call in progress is a handoff call.

All the cells have a channel reservation parameter T_m for traffic of class m. The reservation parameters are intended to give priority to handoff calls with respect to new calls through a reservation policy as follows. Let $\mathbf{n} = (n_1, n_2, \ldots, n_M)$ be a state of cell i where n_m is the number of calls present in cell i of class m. Denote by Ω_i the set of feasible states for cell i, i.e., the set of states $\mathbf{n}$ such that $\sum_{m=1}^{M} n_m b_m \leq C_i$. Let $f_i^{(m)}(\mathbf{n})$ be 1 if cell i in state $\mathbf{n}$ can accept one more call arrival of class m and 0 otherwise, i.e.,

$$f_i^{(m)}(\mathbf{n}) = \begin{cases} 1, & \text{if } C_i - \sum_{k=1}^{M} n_k b_k \geq b_m, \\ 0, & otherwise. \end{cases} \tag{7.1}$$

Define $\mathcal{U}_i^{(m)}$ and $\mathcal{Q}_i^{(m)}$ as the set of *unreserved* and *reserved* states for traffic class m in cell i, respectively, where

$$\mathcal{U}_i^{(m)} = \left\{ \mathbf{n} : T_m < \left\lfloor \frac{C_i - \sum_{k=1}^{M} n_k b_k}{b_m} \right\rfloor \right\}, \quad \mathcal{Q}_i^{(m)} = \left\{ \mathbf{n} : T_m \geq \left\lfloor \frac{C_i - \sum_{k=1}^{M} n_k b_k}{b_m} \right\rfloor \right\}. \tag{7.2}$$

Define the set of *blocked* states for traffic class m in cell i as

$$\mathcal{B}_i^{(m)} = \left\{ \mathbf{n} : C_i - \sum_{k=1}^{M} n_k b_k < b_m \right\}, \tag{7.3}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. Clearly $\Omega_i = \cup_m \left(\mathcal{U}_i^{(m)} \cup \mathcal{Q}_i^{(m)}\right)$ and $\mathcal{B}_i^{(m)} \subset \mathcal{Q}_i^{(m)}$. The reservation policy can be stated as follows. If a new call of class m arrives to cell i, it is accomodated if the state of cell i, $\mathbf{n}$, is in $\mathcal{U}_i^{(m)}$, otherwise, it is blocked. If a handoff call of class m arrives to cell i, it is blocked only if the state of cell i, $\mathbf{n}$, is in $\mathcal{B}_i^{(m)}$.

We consider that occupancy of the cells evolves according to an M-dimensional birth-death process independent of other cells, where the arrival rate or offered traffic to cell i of class m is $\rho_{i,m}$ for the unreserved states and $\alpha_{i,m}$ for reserved states, and the departure rate when cell i is in state $\mathbf{n}$ for calls of class m is $n_m \mu_{i,m}$. The birth-death process for a cell with two classes of customers, i.e., $M = 2$, can be seen in Figure 7.1.

The sets of reserved states for both traffic classes and the transition rates for state $(1, 2)$ are also shown in the figure. Let $p_i(\mathbf{n})$ be the stationary probability

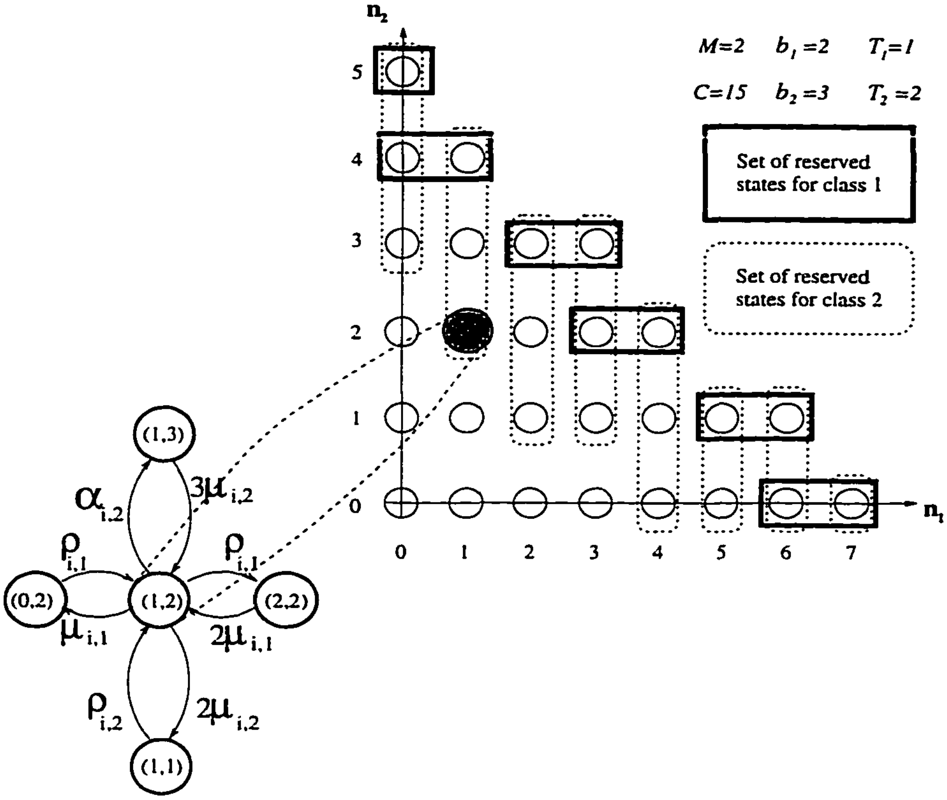

Figure 7.1: Birth-Death Process for a Two-Class Cell Network

that cell i is in state $\mathbf{n}$. Define $\mathcal{I}_{\{D\}}$ as one if event D is true and zero otherwise. From the K-dimensional Markov chain using the global balance equations for cell i in state $\mathbf{n} \in \Omega_i$, we can obtain $p_i(\mathbf{n})$ as follows

$$
\begin{aligned}
p_i(\mathbf{n}) \;=\; & \left\{ \sum_{m=1}^{M} n_m \mu_{i,m} + \sum_{m=1}^{M} f_i^{(m)}(\mathbf{n}) \left[\rho_{i,m} \mathcal{I}_{\{\mathbf{n} \in \mathcal{U}_i^{(m)}\}} + \alpha_{i,m} \mathcal{I}_{\{\mathbf{n} \in \mathcal{Q}_i^{(m)}\}} \right] \right\}^{-1} \\
& \cdot \left\{ \sum_{m=1}^{M} \left[\mathcal{I}_{\{n_m > 0\}} \left(\rho_{i,m} \mathcal{I}_{\{\mathbf{n}_m^- \in \mathcal{U}_i^{(m)}\}} + \alpha_{i,m} \mathcal{I}_{\{\mathbf{n}_m^- \in \mathcal{Q}_i^{(m)}\}} \right) p_i(\mathbf{n}_m^-) \right] \right. \\
& \left. + \sum_{m=1}^{M} f_i^{(m)}(\mathbf{n})(n_m + 1)\mu_{i,m} p_i(\mathbf{n}_m^+) \right\},
\end{aligned}
\tag{7.4}
$$

where the distribution for the states of cell i must satisfy $\sum_{\mathbf{n} \in \Omega_i} p_i(\mathbf{n}) = 1$, and where $\mathbf{n}_i^+ = (n_1, n_2, \ldots, n_i + 1, \ldots, n_M)$ and $\mathbf{n}_i^- = (n_1, n_2, \ldots, n_i - 1, \ldots, n_M)$.

Let $\nu_{ji,m}$ be the handoff rate of class m out of cell j offered to cell i, for adjacent cells i and j. The handoff traffic that can be offered from cell j to an adjacent cell i of class m depends on the proportion of new calls accepted of class m in cell j that goes into cell i, i.e., $\lambda_{j,m}(1 - B_{j,m})q^{(1)}_{ji,m}$, and the proportion of handoff calls accepted of class m from cells adjacent to cell j that goes into cell i, i.e., $(1 - B_{hj,m})q^{(2)}_{ji,m} \sum_{x \in \mathcal{A}_j} \nu_{xj,m}$. Thus, the handoff rate out of cell j offered to cell i of traffic class m is given by

$$\nu_{ji,m} = \lambda_{j,m}(1 - B_{j,m})q^{(1)}_{ji,m} + (1 - B_{hj,m})q^{(2)}_{ji,m} \sum_{x \in \mathcal{A}_j} \nu_{xj,m} \ . \tag{7.5}$$

Figure 7.2 contains a representation of the terms involved in equation (7.5) where only cells adjacent to cell i are shown and the subindex for the class is dropped.

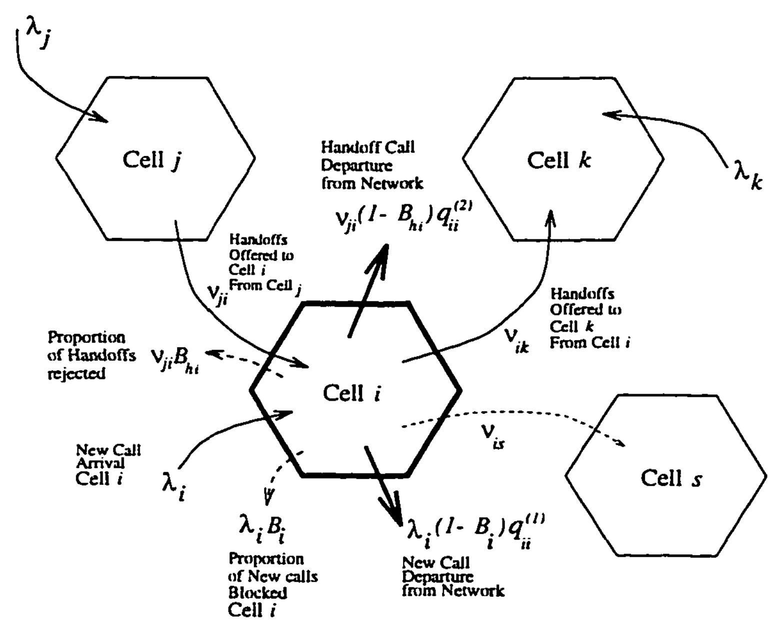

Figure 7.2: Handoff Rate Offered from Adjacent Cells (single rate case)

This set of linear simultaneous equations in $\nu_{ji,m}$ can be solved to compute the total offered traffic of class m to cell i. This is given by

$$\rho_{i,m} = \lambda_{i,m} + \sum_{j \in \mathcal{A}_i} \nu_{ji,m} \ , \quad \mathrm{n} \in \mathcal{U}_i^{(m)}, \tag{7.6}$$

$$\alpha_{i,m} = \sum_{j \in \mathcal{A}_i} \nu_{ji,m} \ , \quad \mathrm{n} \in \mathcal{Q}_i^{(m)} \setminus \mathcal{B}_i^{(m)}. \tag{7.7}$$

For $\mathbf{n} \in \mathcal{B}_i^{(m)}$ the total offered traffic is zero. In Appendix A.1, it is shown that equations (7.5), (7.6) and (7.7) have a unique solution ν_{ji}, ρ_i and α_i for given values of blocking probabilities.

The new call blocking probability for class m in cell i, $B_{i,m}$, and the handoff drop probability for class m in cell i, $B_{hi,m}$, are given as follows:

$$B_{i,m} = \sum_{\mathbf{r} \in \mathcal{Q}_i^{(m)}} p_i(\mathbf{r}), \quad \text{and} \quad B_{hi,m} = \sum_{\mathbf{r} \in \mathcal{B}_i^{(m)}} p_i(\mathbf{r}). \tag{7.8}$$

Given that a call of traffic class m has been originated in a particular cell, say cell j, we define the forced termination probability [53], $B_{j,m}^d$, as the probability that the call is terminated due to a handoff failure during its lifetime. The expression for $B_{j,m}^d$ is obtained in Appendix A.2.

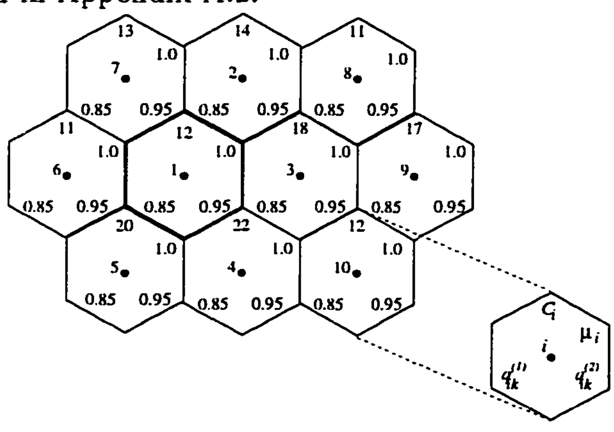

Figure 7.3: Ten-Cell Network Used in Examples with Parameters for Single Rate Case

The calculations and simulations were done for the 10-cell network shown in Figure 7.3. In the figures, $N = 10$ refers to the network with 10 cells, $L = 25\%$ to the total new call arrival rate as explained, bw1(=1) and bw2(=2) to the number of channels a call of class 1 or class 2 uses, respectively, $T = [x, y]$ refers to the reservation of x channels for calls of class 1 and y channels for class 2 in all the cells

Table 7.1: Parameter for a Ten-Cell Network at 25% Load, ($\lambda_{1,1}$ was varied, two classes)

Simulation Parameters									
Cell i	$\lambda_{i,1}$	$\lambda_{i,2}$	C_i	$\mu_{i,1}$	$\mu_{i,2}$	$q_{ii,1}^{(1)}$	$q_{ii,1}^{(2)}$	$q_{ii,2}^{(1)}$	$q_{ii,2}^{(2)}$
1	*	0.1363	12	1.0	0.5	0.85	0.95	0.90	0.97
2	3.1818	0.1590	14	1.0	0.5	0.85	0.95	0.90	0.97
3	4.0909	0.2045	18	1.0	0.5	0.85	0.95	0.90	0.97
4	5.0000	0.2500	22	1.0	0.5	0.85	0.95	0.90	0.97
5	4.5454	0.2272	20	1.0	0.5	0.85	0.95	0.90	0.97
6	2.5000	0.1250	11	1.0	0.5	0.85	0.95	0.90	0.97
7	2.9545	0.1477	13	1.0	0.5	0.85	0.95	0.90	0.97
8	2.5000	0.1250	11	1.0	0.5	0.85	0.95	0.90	0.97
9	3.8636	0.1931	17	1.0	0.5	0.85	0.95	0.90	0.97
10	2.7272	0.1363	12	1.0	0.5	0.85	0.95	0.90	0.97

of the network, and $\mu = [x, y]$ refers to a dwell time of $1/x$ for calls of class 1 and $1/y$ for class 2 in all the cells. For traffic class 2 the mean dwell time on each cell is of 2 time units and for class 1 is of one time unit. The departure probabilities for class 1, $q_{ii,1}^{(1)}$ and $q_{ii,1}^{(2)}$, were chosen to be 0.85 and 0.95, respectively, and for class 2, $q_{ii,2}^{(1)}$ and $q_{ii,2}^{(2)}$ were 0.9 and 0.97. The parameters are summarized in Table 7.1. In the examples, the channel reservation parameters of the classes were varied and the performance evaluated in terms of the new call blocking probability, $B_{i,m}$, and the handoff drop probability, $B_{hi,m}$, of both classes. The figures shown were obtained by keeping the new call arrival for all the cells and both classes constant except for class 1 of cell 1 since this is one of the cells with the most number of adjacent cells. The total new call arrival rate into a cell for both classes is of 25% of the capacity of that cell, e.g., for cell 10 with capacity 12 the total arrival rate is 2.7272+2(0.1363)=3.0 which is 25% of 12.

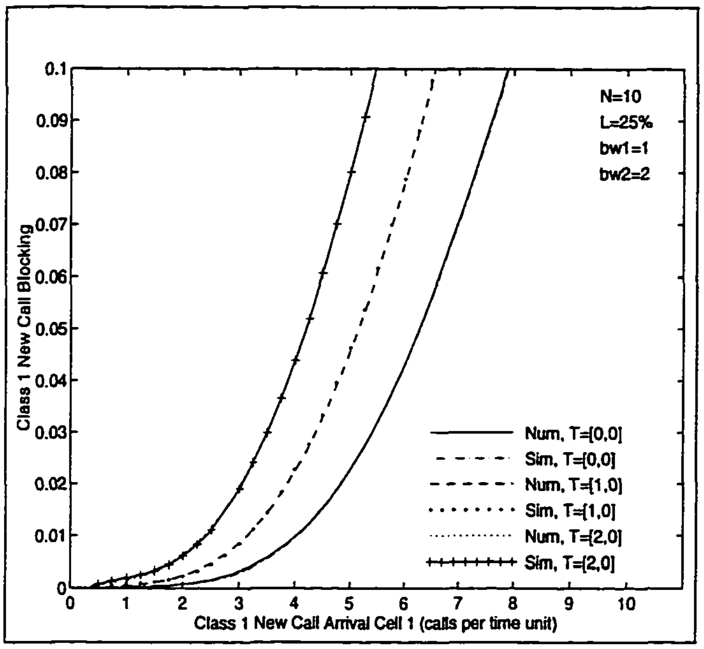

Figure 7.4: Class 1 New Call Blocking of Cell 1 for 10-Cell Network, $\mu = [1, \frac{1}{2}]$

Figures 7.4 - 7.6 contain graphs of comparison of the numerical results for new call blocking and handoff blocking with the simulations for different channel reservation parameters. In all cases it can be seen that simulation results are extremely close to the numerical results, thereby validating the fixed point model. It can be seen that for class 1 calls, the new call blocking for the case of no reservation performs better than the cases with reservation, whereas for handoff drop probability, when channels are reserved for class 1, the performance improves. This improvement is due to the priority given to handoffs by reserving channels in each cell. In all the cases shown, there is no channel reservation for class 2, hence, the new call blocking and the handoff drop probabilities are the same for class 2, with better performance

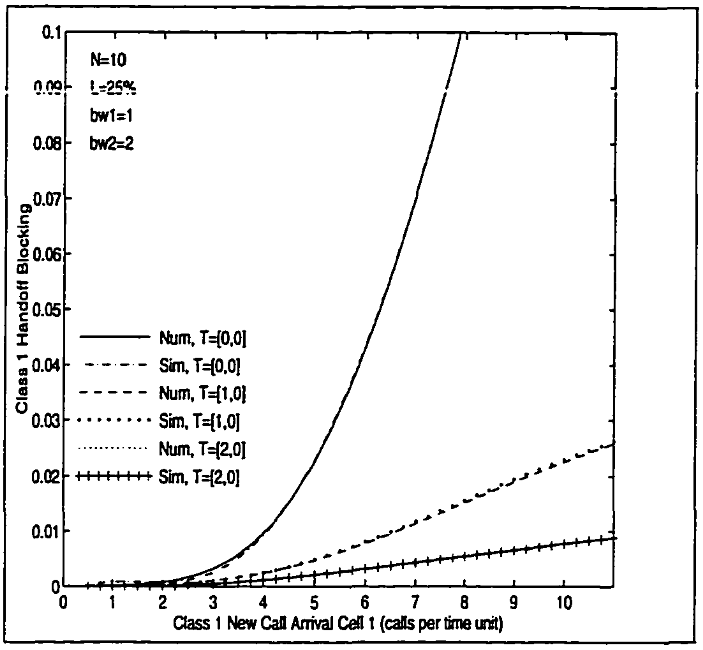

Figure 7.5: Class 1 Handoff Blocking of Cell 1 for 10-Cell Network, $\mu = [1, \frac{1}{2}]$

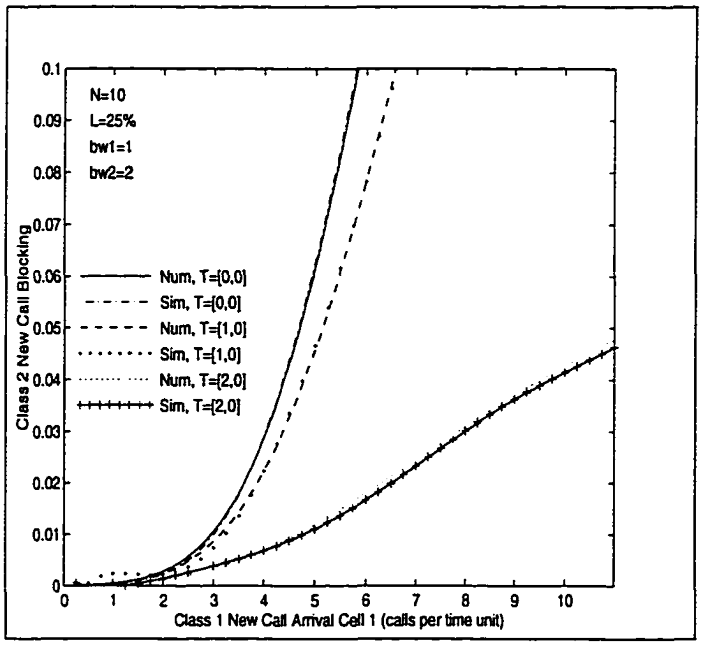

Figure 7.6: Class 2 New Call Blocking of Cell 1 for 10-Cell Network, $\mu = [1, \frac{1}{2}]$

when there are channels reserved for handoffs of class 1, i.e., more new calls of class 1 will be rejected but since there is no reservation for class 2, more new calls or handoff calls of class 2 will be accepted.

In the following section, we define the network net revenue and formulate the shadow price methodology for the FCA model to be used in the numerical results.

7.3 Shadow Prices for Multirate Wireless Networks

Define the net revenue, W, as the revenue generated by the traffic which is carried succesfully. This revenue consists of two components: the first one is the revenue generated by accepting in each cell j a new call of class m, $\lambda_{j,m}\,(1-B_{j,m})\,w_{j,m}$, the second takes into account the cost of a forced termination due to handoff failure of those new calls of class m that have arrived and been accepted in cell j, i.e., $\lambda_{j,m}\,(1-B_{j,m})\,c_{j,m}B^d_{j,m}$. Hence the net revenue is

$$W(\mathbf{B},\mathbf{B}^d,\underline{\lambda}) = \sum_{m=1}^{M}\sum_{j\in\mathcal{N}} \lambda_{j,m}\,(1-B_{j,m}(\underline{\lambda},\mathbf{p}))\left\{w_{j,m} - c_{j,m}B^d_{j,m}(\underline{\lambda},\mathbf{p})\right\}, \qquad (7.9)$$

where $w_{j,m}$ is the revenue generated by accepting a call of class m in cell j, and $c_{j,m}$ is the cost of a forced termination of a call of class m due to a handoff failure. In the *Appendix* A.3, it is shown that (7.9) can be rewritten in the form

$$\begin{aligned} W(\mathbf{B},\mathbf{B}_h,\underline{\lambda},\underline{\alpha}) &= \sum_{m=1}^{M}\sum_{i\in\mathcal{N}}\Big\{w_{i,m}\lambda_{i,m}\,(1-B_{i,m}(\underline{\lambda},\mathbf{p})) \\ &- c_{i,m}B_{hi,m}(\underline{\lambda},\mathbf{p})\left\{\mathcal{I}_{\{T_m>0\}}\alpha_{i,m}(\mathbf{v})\right. \\ &+ \left.\mathcal{I}_{\{T_m=0\}}\left[\rho_{i,m}(\underline{\lambda},\mathbf{v})-\lambda_{i,m}\right]\right\}\Big\}, \qquad (7.10) \end{aligned}$$

where $\mathbf{v}$ denotes the vector whose components are the handoff rates $\nu_{ji,m}$ for all $i, j \in \mathcal{N}$ and for $m = 1, 2, \ldots, M$, i.e., $\mathbf{v} = (\nu_{12,1}, \nu_{13,1}, \ldots, \nu_{12,2}, \ldots, \nu_{ji,m}, \ldots)$, $\mathbf{p}$ denotes the vector whose components are the stationary probabilities for each state of all the cells, $\mathbf{B}$, the vector of the new call blocking probabilities for all the cells and classes, $\mathbf{B}_h$, the vector of the handoff call blocking probabilities for all the cells and classes, $\underline{\lambda}$, the vector of new call arrival rates and $\underline{\alpha}$, the vector of offered traffic to the cells for both, reserved and unreserved states. $B_{i,s}(\underline{\lambda}, \mathbf{p})$ and $B_{hi,s}(\underline{\lambda}, \mathbf{p})$ are given by equation (7.8) and are written here to explicitly show their dependence on $\underline{\lambda}$ and $\mathbf{p}$. The shadow price calculation of equation (7.10) is less complicated than that of (7.9) since the latter requires one more derivative to calculate because the forced termination probability is a function of the handoff blocking probabilities, $\mathbf{B}_h$. The second term in (7.10) is the cost due to the proportion of rejected handoff attempts in cell i.

The choice of the revenue for cell i and traffic class m, $w_{i,m}$, depends on the number of channels each call of that class occupies, b_m, and the average holding time of the calls of that class. The average holding time depends on the average number of handoffs the calls undergo before departure from the network and since every time a call is accepted in a cell its duration in that cell is the dwell time with mean $1/\mu_{i,m}$, the average holding time of a call will be given by the average number of handoffs times the dwell time.

Therefore, in (7.9) and (7.10), we take the revenues to be $w_{i,m} = b_m \cdot$(average holding time). Note that since the equilibrium probability vectors and blocking probabilities we use are those given by the fixed point model, equations (7.4) through (7.8), rather than the actual equilibrium probabilities and blocking probabilities, the revenue W in (7.10) is an *approximate* network net revenue. As a result, the shadow prices we calculate are also approximations to the actual shadow prices. However, in Section 7.3.1, we present the evaluation of shadow prices from simulations which verify the accuracy of the approximation.

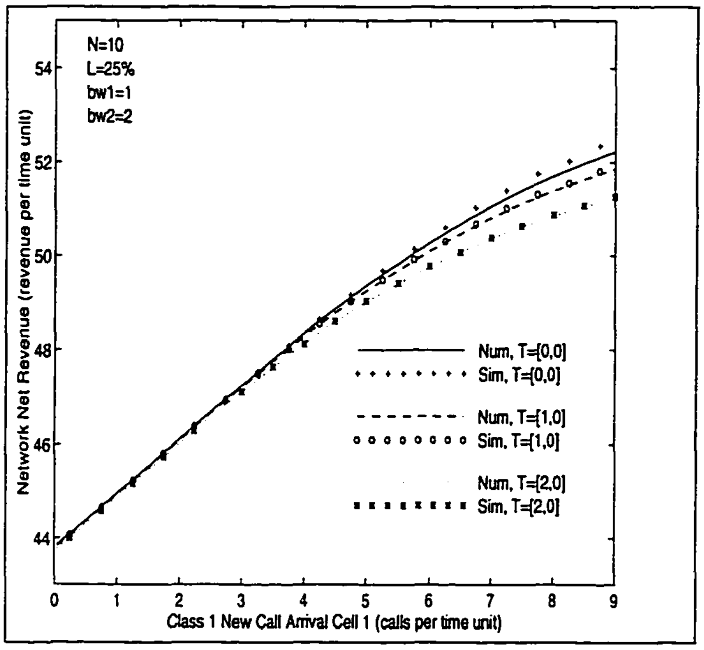

Figure 7.7: Net Revenue for 10-Cell Network, $\mu = [1, \frac{1}{2}]$

Figure 7.7 compares the network net revenue as a function of the new call arrival of cell 1 for the 10-cell network obtained from simulation and from the fixed point model. The revenue generated by accepting a call of class one in cell i, i.e., $w_{i,1}$, that is chosen according to the average holding time is 1.1579 for class 1 and 4.4124 for class 2 for all the cells. The cost of rejecting a class 1 call in cell i, i.e., $c_{i,1}$, was set to 2.1579, and the cost of rejecting a class 2 call in cell i was set to 5.4124, i.e., one unit more than the corresponding revenue. The figure provides the net revenue for the same cases of network parameters as those in figures 7.4 - 7.6. It can be seen that as the channel reservation for calls of class 1 increases, the net revenue decreases due to the increase in new call blocking as shown in Figure 7.4.

We also analyzed the single rate case for the 10-cell network with low and high mobility, where for low mobility the departure probabilities for new calls, $q^{(1)}_{ii,m}$, and handoff calls, $q^{(2)}_{ii,m}$, are 0.85 and 0.95, respectively, and for high mobility these probabilities are 0.35 and 0.6. To analyze the effect of mobility and overload, we considered the new call arrival for each cell at 50% of its corresponding capacity and called this our base demand. The base demand corresponds to the case of 0% overload. The 100% overload has twice as much new call arrival rate into each cell as the base demand.

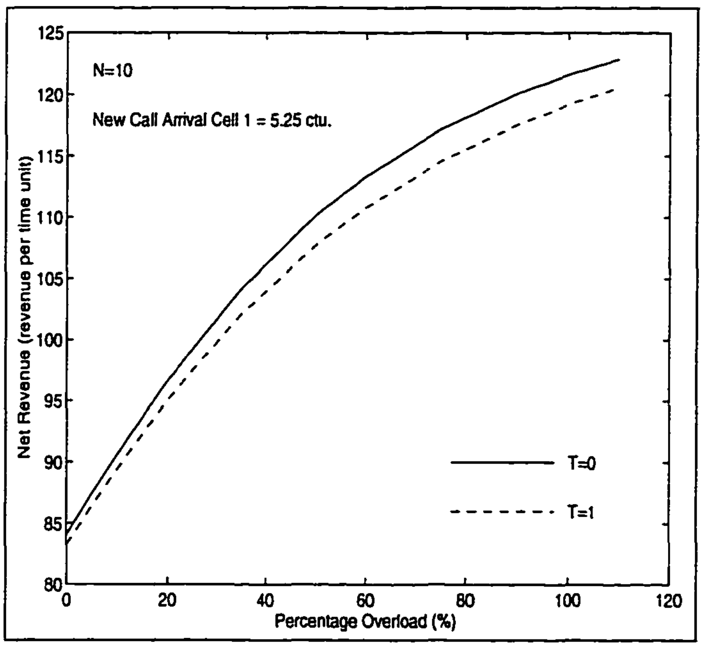

Figure 7.8: Net Revenue of Ten-Cell Network (single rate, low mobility)

We increased the demand from 0% to 100% overload and calculated the net revenue for the 10-cell network single rate case. We varied the external arrival of cell 1 keeping constant the remaining arrival rates at the overload percentage used. The calculation was done for reservation parameter for all the cells kept at $T = 0$ and $T = 1$. The revenues, w_i, were set at 2.0833, and the costs, c_i, at 3.0833. Figure 7.8 shows the net revenue, W, for the ten-cell network versus overload at a fixed new call arrival for cell 1 of 5.25 calls per time unit (ctu) for the low mobility case. It can be seen that with low mobility the reservation of channels decreases revenue because new call blocking is increased and the number of handoffs is too small to get benefit from reserving channels. Figure 7.9 shows the surface for the revenue but for the high mobility case. The net revenue versus overload and new call arrival of cell 1 is shown. It can be seen that for the high mobility case, the use of reservation improves performance over the case of no reservation due to the increase on the number of handoffs in the network that are given priority by reserving one channel and that would be rejected otherwise. Figure 7.10 shows the net revenue for the 10-cell network in the single rate case as a function of percentage overload with the new call arrival rate of cell 1 kept constant at 5.25 calls per time unit (ctu). It can be seen that for overload cases reserving channels to reduce handoff drop improves significantly the network revenue performance over the no reservation case.

The fixed point model describes the $\mathbf{p}$ as an implicit function of $\underline{\lambda}$. $\mathbf{B}$ and $\mathbf{B}_h$ are, in turn, functions of $\mathbf{p}$ and thereby implicit functions of $\underline{\lambda}$. Consequently, $W(\mathbf{B}, \mathbf{B}_h, \underline{\lambda}, \underline{\alpha})$ is also an implicit function of $\underline{\lambda}$. We therefore undertake a careful and extensive effort to obtain relations of total and partial derivatives of the new call and handoff blocking probabilities by differentiating the fixed point equations. These

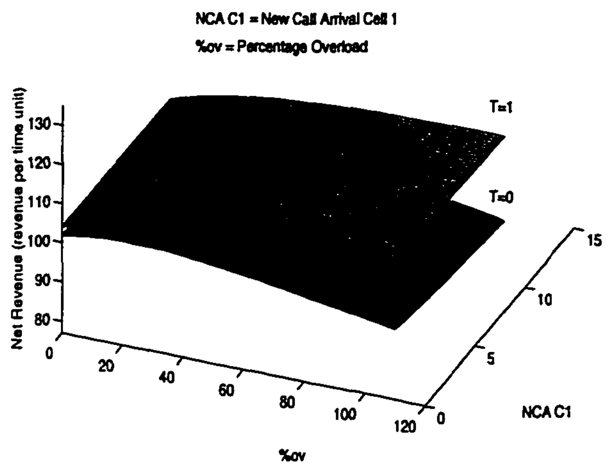

Figure 7.9: Net Revenue of Ten-Cell Network (single rate)

relations are manipulated to obtain a system of linear equations in the derivatives of the new call and handoff blocking probabilities with respect to new call traffic.

7.3.1 Calculation of the Shadow Price

We first need to define the total derivative of the net revenue function with respect to new call arrival rates as follows

$$\begin{aligned}\frac{dW(\mathbf{B},\mathbf{B}_h,\underline{\lambda},\underline{\alpha})}{d\lambda_{k,m}} &= \frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\lambda},\underline{\alpha})}{\partial\lambda_{k,m}} \\ &+\sum_{s=1}^{M}\sum_{i\in\mathcal{N}}\left\{\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\lambda},\underline{\alpha})}{\partial B_{i,s}}\frac{dB_{i,s}(\underline{\lambda},\mathbf{p})}{d\lambda_{k,m}}\right. \\ &+\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\lambda},\underline{\alpha})}{\partial B_{hi,s}}\frac{dB_{hi,s}(\underline{\lambda},\mathbf{p})}{d\lambda_{k,m}}+\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\lambda},\underline{\alpha})}{\partial \alpha_{i,s}}\frac{d\alpha_{i,s}(\mathbf{v})}{d\lambda_{k,m}} \\ &\left.+\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\lambda},\underline{\alpha})}{\partial \rho_{i,s}}\frac{d\rho_{i,s}(\underline{\lambda},\mathbf{v})}{d\lambda_{k,m}}\right\}.\end{aligned} \tag{7.11}$$

The first term in equation (7.11) is obtained as follows:

$$\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\lambda},\underline{\alpha})}{\partial\lambda_{k,m}} = w_{k,m}\left(1-B_{k,m}(\underline{\lambda},\mathbf{p})\right)+\mathcal{I}_{\{T_m=0\}}c_{k,m}B_{hk,m}(\underline{\lambda},\mathbf{p}). \tag{7.12}$$

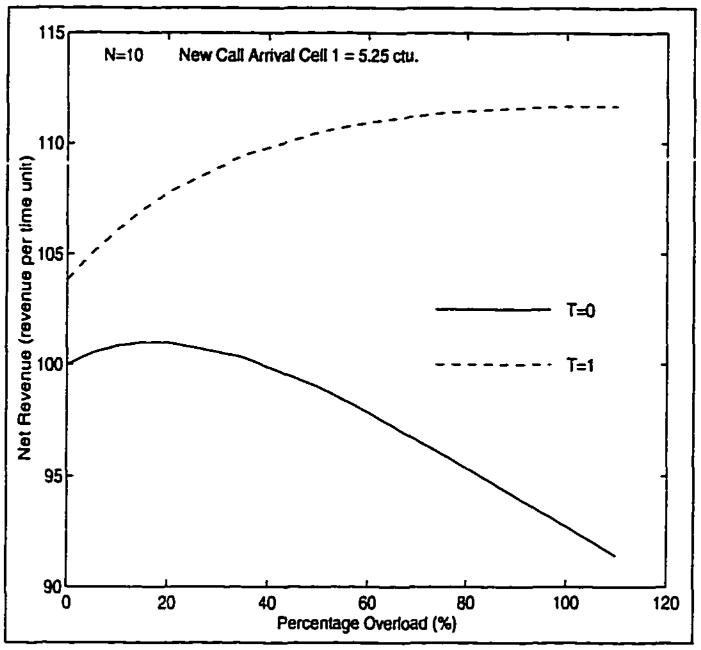

Figure 7.10: Net Revenue of Ten-Cell Network as Overload Increases (single rate, high mobility)

The partial derivatives needed in the shadow price calculations in equation (7.11) are obtained from (7.10) as follows:

$$\frac{\partial W(\mathbf{B}, \mathbf{B}_h, \underline{\lambda}, \underline{\alpha})}{\partial B_{i,s}} = -w_{i,s}\lambda_{i,s}, \tag{7.13}$$

$$\frac{\partial W(\mathbf{B}, \mathbf{B}_h, \underline{\lambda}, \underline{\alpha})}{\partial B_{hi,s}} = -c_{i,s}\left\{\mathcal{I}_{\{T_s>0\}}\alpha_{i,s}(\mathbf{v}) + \mathcal{I}_{\{T_s=0\}}\left[\rho_{i,s}(\underline{\lambda}, \mathbf{v}) - \lambda_{i,s}\right]\right\}, \tag{7.14}$$

$$\frac{\partial W(\mathbf{B}, \mathbf{B}_h, \underline{\lambda}, \underline{\alpha})}{\partial \alpha_{i,s}} = -c_{i,s}B_{hi,s}(\underline{\lambda}, \mathbf{p})\mathcal{I}_{\{T_s>0\}}, \tag{7.15}$$

$$\frac{\partial W(\mathbf{B}, \mathbf{B}_h, \underline{\lambda}, \underline{\alpha})}{\partial \rho_{i,s}} = -c_{i,s}B_{hi,s}(\underline{\lambda}, \mathbf{p})\mathcal{I}_{\{T_s=0\}}. \tag{7.16}$$

Define $\underline{\alpha}_i$ as the vector of offered traffic of class $s = 1, \ldots, M$ to cell i, i.e., $\underline{\alpha}_i = (\rho_{i,1}, \ldots, \rho_{i,M}, \alpha_{i,1}, \ldots, \alpha_{i,M})$, given in (7.6) and (7.7). Also, $p_i(\underline{\alpha}_i, \mathbf{n})$ are given by (7.4) and written here to show explicitly their dependence on $\underline{\alpha}_i$ and $\mathbf{n}$. From equation (7.8) we get

$$\frac{dB_{hi,s}(\underline{\lambda}, \mathbf{p})}{d\lambda_{k,m}} = \sum_{\mathbf{n}\in\mathcal{B}_i^{(m)}} \frac{dp_i(\underline{\alpha}_i, \mathbf{n})}{d\lambda_{k,m}}, \tag{7.17}$$

$$\frac{dB_{i,s}(\underline{\lambda},\mathbf{p})}{d\lambda_{k,m}} = \sum_{\mathbf{n}\in\mathcal{Q}_i^{(m)}} \frac{dp_i(\underline{\alpha}_i,\mathbf{n})}{d\lambda_{k,m}}. \tag{7.18}$$

Now, the total derivatives needed in (7.17) and (7.18) can be obtained as follows

$$\frac{dp_i(\underline{\alpha}_i,\mathbf{n})}{d\lambda_{k,m}} = \sum_{s=1}^{M} \frac{\partial p_i(\underline{\alpha}_i,\mathbf{n})}{\partial \rho_{i,s}} \frac{d\rho_{i,s}(\underline{\lambda},\mathbf{v})}{d\lambda_{k,m}} + \sum_{s=1}^{M} \left(\mathcal{I}_{\{T_s>0\}}\right) \frac{\partial p_i(\underline{\alpha}_i,\mathbf{n})}{\partial \alpha_{i,s}} \frac{d\alpha_{i,s}(\mathbf{v})}{d\lambda_{k,m}}. \tag{7.19}$$

Equation (7.19) requires the partial derivative of the distribution for the states of cell i with respect to the offered traffic of class s to this cell in the unreserved and reserved states. Fix $\mathbf{n}$ as the state of cell i. If $\mathbf{n}$ is an unreserved state, i.e., $\mathbf{n} \in \mathcal{U}_i^{(s)}$, from equation (7.4), the set of global balance equations obtained from the M-dimensional Markov chain, and the normalizing equation $\sum_{\mathbf{n}\in\Omega_i} p_i(\underline{\alpha}_i,\mathbf{n}) = 1$ we can obtain the partial derivatives as follows.

$$\begin{aligned}
\frac{\partial p_i(\underline{\alpha}_i,\mathbf{n})}{\partial \rho_{i,s}} = {} & \Bigg\{ \sum_{m=1}^{M} n_m \mu_{i,m} + \sum_{m=1}^{M} f_i^{(m)}(\mathbf{n}) \Big[\rho_{i,m}(\underline{\lambda},\mathbf{v}) \mathcal{I}_{\{\mathbf{n}\in\mathcal{U}_i^{(m)}\}} \\
& + \alpha_{i,m}(\mathbf{v}) \mathcal{I}_{\{\mathbf{n}\in\mathcal{Q}_i^{(m)}\}} \Big] \Bigg\}^{-2} \\
& \Bigg(\left[\sum_{m=1}^{M} n_m \mu_{i,m} + \sum_{m=1}^{M} f_i^{(m)}(\mathbf{n}) \left(\rho_{i,m}(\underline{\lambda},\mathbf{v}) \mathcal{I}_{\{\mathbf{n}\in\mathcal{U}_i^{(m)}\}} + \alpha_{i,m}(\mathbf{v}) \mathcal{I}_{\{\mathbf{n}\in\mathcal{Q}_i^{(m)}\}} \right) \right] \\
& \cdot \Bigg[\sum_{m=1}^{M} \mathcal{I}_{\{n_m>0\}} \rho_{i,m}(\underline{\lambda},\mathbf{v}) \mathcal{I}_{\{\mathbf{n}_m^-\in\mathcal{U}_i^{(m)}\}} \frac{\partial p_i(\underline{\alpha}_i,\mathbf{n}_i^-)}{\partial \rho_{i,s}} \\
& + \mathcal{I}_{\{n_s>0\}} \mathcal{I}_{\{\mathbf{n}_s^-\in\mathcal{U}_i^{(s)}\}} p_i(\underline{\alpha}_i,\mathbf{n}_i^-) \\
& + \sum_{m=1}^{M} f_i^{(m)}(\mathbf{n})(n_m+1)\mu_{i,m} \frac{\partial p_i(\underline{\alpha}_i,\mathbf{n}_m^+)}{\partial \rho_{i,s}} \Bigg] \\
& - \mathcal{I}_{\{\mathbf{n}\in\mathcal{U}_i^{(s)}\}} f_i^{(s)}(\mathbf{n}) \Bigg\{ \sum_{m=1}^{M} \Big[\mathcal{I}_{\{n_m>0\}} \Big(\rho_{i,m}(\underline{\lambda},\mathbf{v}) \mathcal{I}_{\{\mathbf{n}_m^-\in\mathcal{U}_i^{(m)}\}} \\
& + \alpha_{i,m}(\mathbf{v}) \mathcal{I}_{\{\mathbf{n}_m^-\in\mathcal{Q}_i^{(m)}\}} \Big) p_i(\underline{\alpha}_i,\mathbf{n}_m^-) \Big] \\
& + \sum_{m=1}^{M} f_i^{(m)}(\mathbf{n})(n_m+1)\mu_{i,m} p_i(\underline{\alpha}_i,\mathbf{n}_m^+) \Bigg\} \Bigg),
\end{aligned} \tag{7.20}$$

and $\sum_{\mathbf{n}\in\Omega_i} \frac{\partial p_i(\underline{\alpha}_i,\mathbf{n})}{\partial \rho_{i,s}} = 0$.

Similarly for the partial derivative with respect to the offered traffic of class s in the reserved states, i.e., $\alpha_{i,s}(\mathbf{v})$ we have

$$\begin{aligned}
\frac{\partial p_i(\underline{\alpha}_i, \mathbf{n})}{\partial \alpha_{i,s}} = & \left\{ \sum_{m=1}^{M} n_m \mu_{i,m} + \sum_{m=1}^{M} f_i^{(m)}(\mathbf{n}) \left[\rho_{i,m}(\underline{\lambda}, \mathbf{v}) \mathcal{I}_{\{\mathbf{n} \in \mathcal{U}_i^{(m)}\}} \right.\right. \\
& \left.\left. + \alpha_{i,m}(\mathbf{v}) \mathcal{I}_{\{\mathbf{n} \in \mathcal{Q}_i^{(m)}\}} \right] \right\}^{-2} \\
& \left(\left[\sum_{m=1}^{M} n_m \mu_{i,m} + \sum_{m=1}^{M} f_i^{(m)}(\mathbf{n}) \left(\rho_{i,m}(\underline{\lambda}, \mathbf{v}) \mathcal{I}_{\{\mathbf{n} \in \mathcal{U}_i^{(m)}\}} + \alpha_{i,m}(\mathbf{v}) \mathcal{I}_{\{\mathbf{n} \in \mathcal{Q}_i^{(m)}\}} \right) \right] \right. \\
& \cdot \left[\sum_{m=1}^{M} \mathcal{I}_{\{n_m > 0\}} \alpha_{i,m}(\mathbf{v}) \mathcal{I}_{\{\mathbf{n}_m^- \in \mathcal{Q}_i^{(m)}\}} \frac{\partial p_i(\underline{\alpha}_i, \mathbf{n}_i^-)}{\partial \alpha_{i,s}} \right. \\
& + \mathcal{I}_{\{n_s > 0\}} \mathcal{I}_{\{\mathbf{n}_s^- \in \mathcal{Q}_i^{(s)}\}} p_i(\underline{\alpha}_i, \mathbf{n}_i^-) \\
& \left. + \sum_{m=1}^{M} f_i^{(m)}(\mathbf{n})(n_m + 1)\mu_{i,m} \frac{\partial p_i(\underline{\alpha}_i, \mathbf{n}_m^+)}{\partial \alpha_{i,s}} \right] \\
& - \mathcal{I}_{\{\mathbf{n} \in \mathcal{Q}_i^{(s)}\}} f_i^{(s)}(\mathbf{n}) \left\{ \sum_{m=1}^{M} \left[\mathcal{I}_{\{n_m > 0\}} \left(\rho_{i,m}(\underline{\lambda}, \mathbf{v}) \mathcal{I}_{\{\mathbf{n}_m^- \in \mathcal{U}_i^{(m)}\}} \right.\right.\right. \\
& \left.\left. + \alpha_{i,m}(\mathbf{v}) \mathcal{I}_{\{\mathbf{n}_m^- \in \mathcal{Q}_i^{(m)}\}} \right) p_i(\underline{\alpha}_i, \mathbf{n}_m^-) \right] \\
& \left.\left. + \sum_{m=1}^{M} f_i^{(m)}(\mathbf{n})(n_m + 1)\mu_{i,m} p_i(\underline{\alpha}_i, \mathbf{n}_m^+) \right\} \right),
\end{aligned} \tag{7.21}$$

and $\sum_{\mathbf{n} \in \Omega_i} \frac{\partial p_i(\underline{\alpha}_i, \mathbf{n})}{\partial \alpha_{i,s}} = 0$. Equations (7.20) - (7.21) result in a system of linear simultaneous equations in $\frac{\partial p_i(\underline{\alpha}_i, \mathbf{n})}{\partial \rho_{i,s}}$ and $\frac{\partial p_i(\underline{\alpha}_i, \mathbf{n})}{\partial \alpha_{i,s}}$, respectively, for every state $\mathbf{n} \in \Omega_i$, and we can solve it to find the partial derivatives.

Equation (7.19) also requires the total derivative of the offered traffic with respect to the new call arrival rates, $\lambda_{k,m}$. This can be obtained from equations (7.6) and (7.7) as follows:

$$\frac{d\rho_{i,s}(\underline{\lambda}, \mathbf{v})}{d\lambda_{k,m}} = \frac{\partial \rho_{i,s}(\underline{\lambda}, \mathbf{v})}{\partial \lambda_{k,m}} + \sum_{x \in \mathcal{A}_i} \frac{\partial \rho_{i,s}(\underline{\lambda}, \mathbf{v})}{\partial \nu_{xi,s}} \frac{d\nu_{xi,s}(\underline{\lambda}, \mathbf{B}, \mathbf{B}_h, \mathbf{v})}{d\lambda_{k,m}}, \tag{7.22}$$

$$\frac{d\alpha_{i,s}(\mathbf{v})}{d\lambda_{k,m}} = \sum_{x \in \mathcal{A}_i} \frac{\partial \alpha_{i,s}(\mathbf{v})}{\partial \nu_{xi,s}} \frac{d\nu_{xi,s}(\underline{\lambda}, \mathbf{B}, \mathbf{B}_h, \mathbf{v})}{d\lambda_{k,m}}, \tag{7.23}$$

where $\frac{\partial \rho_{i,s}(\underline{\lambda},\mathbf{p})}{\partial \lambda_{k,m}} = \mathcal{I}_{\{i=k,s=m\}}$. The terms $\nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})$ are given by (7.5) and are written here to show their dependence on $\underline{\lambda}$, $\mathbf{B}$, $\mathbf{B}_h$ and $\mathbf{v}$. To continue with the calculation of the shadow prices, we need the partial derivatives of the offered traffic with respect to the handoff rates. From (7.6) or (7.7) we obtain $\frac{\partial \rho_{i,s}(\underline{\lambda},\mathbf{v})}{\partial \nu_{xi,s}} = \mathcal{I}_{\{x\in\mathcal{A}_i\}} = \frac{\partial \alpha_{i,s}(\mathbf{v})}{\partial \nu_{xi,s}}$. Finally from (7.5), the derivatives of the handoff rates with respect to the new call arrivals are given by

$$\begin{aligned}\frac{d\nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{d\lambda_{k,m}} &= \frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial \lambda_{k,m}} \\ &+ \sum_{y\in\mathcal{N}}\left[\frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{y,s}}\frac{dB_{y,s}(\underline{\lambda},\mathbf{p})}{d\lambda_{k,m}}\right. \\ &+ \left.\frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy,s}}\frac{dB_{hy,s}(\underline{\lambda},\mathbf{p})}{d\lambda_{k,m}}\right].\end{aligned} \tag{7.24}$$

The partial derivatives needed in (7.24) are obtained by the following

$$\frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial \lambda_{k,m}} = \mathcal{I}_{\{k=x,s=m\}}\left(1 - B_{x,s}(\underline{\lambda},\mathbf{p})\right) q^{(1)}_{xi,s}. \tag{7.25}$$

From the set of simultaneous equations in (7.5) we can obtain the following expression for the partial derivative of the handoff rates, $\nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})$, with respect to the new call blocking.

$$\begin{aligned}\frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{y,s}} &= -\lambda_{x,s} q^{(1)}_{xi,s}\mathcal{I}_{\{x=y\}} \\ &+ \left(1 - B_{hx,s}(\underline{\lambda},\mathbf{p})\right) q^{(2)}_{xi,s}\sum_{z\in\mathcal{A}_x}\frac{\partial \nu_{zx,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{y,s}},\end{aligned} \tag{7.26}$$

where $\mathcal{I}_{\{D\}}$ is equal to one if event D is true and zero otherwise. We can see that (7.26) is also a set of simultaneous equations which can be solved for $\frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{y,s}}$. Similarly, for the partial derivative of the handoff rates of class s, $\nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})$, with respect to the handoff blocking probability, $B_{hx,s}(\underline{\lambda},\mathbf{p})$, from (7.5) we obtain

$$\frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy,s}} = (1 - B_{hx,s}(\underline{\lambda},\mathbf{p}))\, q^{(2)}_{xi,s} \sum_{z\in A_x} \frac{\partial \nu_{zx,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy,s}} - \mathcal{I}_{\{x=y\}}\, q^{(2)}_{xi,s} \sum_{z\in A_x} \nu_{zx,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v}), \tag{7.27}$$

which is a set of simultaneous equations which can be solved for $\frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy,s}}$. Substituting (7.22) and (7.23) in (7.19) and this into (7.17) and (7.18) results in expressions for the total derivatives $\frac{dB_{y,s}(\underline{\lambda},\mathbf{p})}{d\lambda_{k,m}}$ and $\frac{dB_{hy,s}(\underline{\lambda},\mathbf{p})}{d\lambda_{k,m}}$ in terms of the total derivatives $\frac{d\nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{d\lambda_{k,m}}$. These expressions can be further substituted in (7.24) to obtain the following system of linear simultaneous equations in $\frac{d\nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{d\lambda_{k,m}}$.

$$\begin{aligned}
\frac{d\nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{d\lambda_{k,m}} &= \frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial \lambda_{k,m}} \\
&+ \sum_{y\in\mathcal{N}} \left[\frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{y,s}} \sum_{\mathbf{n}\in\mathcal{Q}_y^{(s)}} \sum_{r=1}^{M} \frac{\partial p_y(\underline{\alpha}_y,\mathbf{n})}{\partial \rho_{y,r}} \frac{\partial \rho_{y,r}(\underline{\lambda},\mathbf{v})}{\partial \lambda_{k,m}} \right. \\
&+ \left. \frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy,s}} \sum_{\mathbf{n}\in\mathcal{B}_y^{(s)}} \sum_{r=1}^{M} \frac{\partial p_y(\underline{\alpha}_y,\mathbf{n})}{\partial \rho_{y,r}} \frac{\partial \rho_{y,r}(\underline{\lambda},\mathbf{v})}{\partial \lambda_{k,m}} \right] \\
&+ \sum_{r=1}^{M} \sum_{y\in\mathcal{N}} \sum_{z\in A_y} \left\{ \sum_{\mathbf{n}\in\mathcal{Q}_y^{(s)}} \frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{y,s}} \right. \\
&\left[\frac{\partial p_y(\underline{\alpha}_y,\mathbf{n})}{\partial \rho_{y,r}} \frac{\partial \rho_{y,r}(\underline{\lambda},\mathbf{v})}{\partial \nu_{zy,r}} + \mathcal{I}_{\{T_s>0\}} \frac{\partial p_y(\underline{\alpha}_y,\mathbf{n})}{\partial \alpha_{y,r}} \frac{\partial \alpha_{y,r}(\mathbf{v})}{\partial \nu_{zy,r}} \right] \\
&+ \frac{\partial \nu_{xi,s}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy,s}} \left[\sum_{\mathbf{n}\in\mathcal{B}_y^{(s)}} \frac{\partial p_y(\underline{\alpha}_y,\mathbf{n})}{\partial \rho_{y,r}} \frac{\partial \rho_{y,r}(\underline{\lambda},\mathbf{v})}{\partial \nu_{zy,r}} \right. \\
&+ \left.\left. \mathcal{I}_{\{T_s>0\}} \sum_{\mathbf{n}\in\mathcal{B}_y^{(s)}} \frac{\partial p_y(\underline{\alpha}_y,\mathbf{n})}{\partial \alpha_{y,r}} \frac{\partial \alpha_{y,r}(\mathbf{v})}{\partial \nu_{zy,r}} \right] \right\} \frac{d\nu_{zy,r}(\underline{\lambda},\mathbf{B},\mathbf{B}_h,\mathbf{v})}{d\lambda_{k,m}}.
\end{aligned} \tag{7.28}$$

The set of simultaneous equations (7.28) is solved and the results used in (7.22), (7.23), (7.19), (7.17) and (7.18) to substitute into (7.11).

In Figures 7.11 and 7.12, we compared the shadow prices obtained from numerical calculation with those obtained from simulation for the 10-cell network as a function

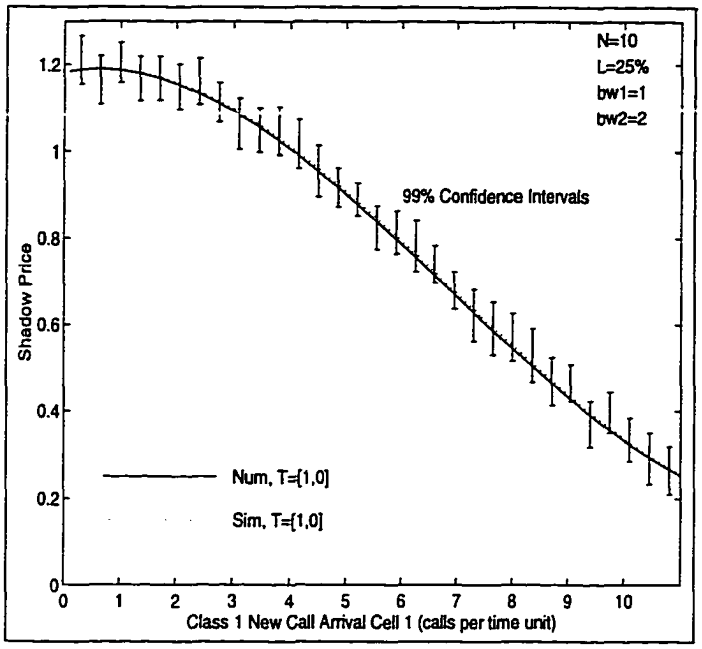

Figure 7.11: Shadow Price of Net Revenue with Respect to Class 1 New Call Arrival of Cell 1 for 10-Cell Network, $\mu = [1, \frac{1}{2}]$

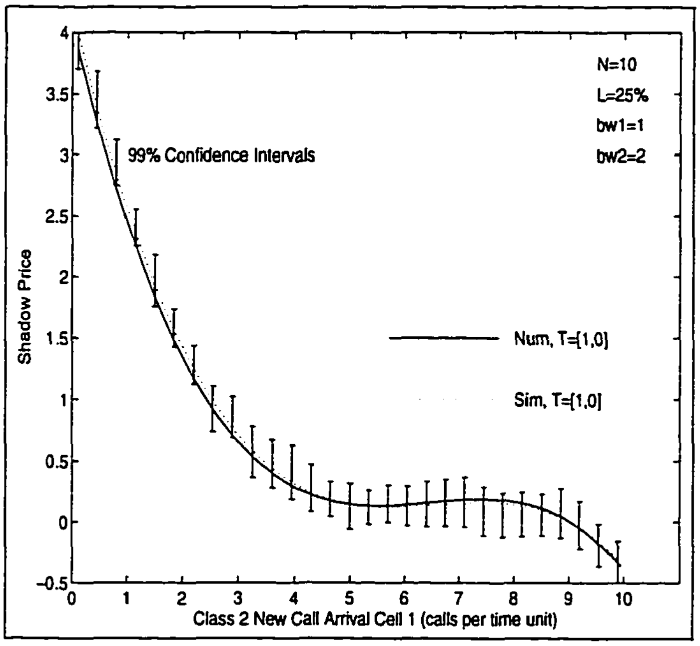

Figure 7.12: Shadow Price of Net Revenue with Respect to Class 2 New Call Arrival of Cell 1 for 10-Cell Network, $\mu = [1, \frac{1}{2}]$

of the new call arrival rate of cell 1 of class 1 and 2, respectively. In Figures 7.11 and 7.12, the shadow price of the net revenue is shown with respect to the new call arrival of class 1 and class 2 of cell 1, respectively, together with the shadow price obtained by simulation as just explained. The 99% confidence intervals are also included to show the close agreement of the model with the simulation. The shadow price with respect to class 2 decreases rapidly because of the degradation on performance by accepting more calls that occupy two channels instead of one, increasing blocking.

7.3.2 Complexity

In this section we investigate the complexity of the fixed point and shadow price algorithm of Section 7.2 for a network with M classes of traffic and with N cells, each one of them with C channels.

Let S be the total number of states in the M-dimensional Markov chain of any of the cells. Given the offered traffic of class s to every cell i, $\rho_{i,s}$ and $\alpha_{i,s}$, the complexity to find the state probabilities of cell i, $p_i(\mathbf{n})$, is that of solving the S linear simultaneous equations. By solving the corresponding matrix equation using Coppersmith and Winograd's algorithm [47], which has complexity $O(n^{2.376})$ for a matrix of size n, it can be seen that the solution of this set of simultaneous equations for all the cells in the network has complexity $O(S^{2.376}N)$.

Knowing the blocking probabilities of each class in every cell, $B_{i,s}$ and $B_{hi,s}$, the handoff rates, $\nu_{ji,s}$, require the solution of δ simultaneous equations for each class on the entire network, where δ is the sum of the number of adjacent cells to each cell in the network, i.e., in terms of graph theory, this is the sum of the *degrees* of all the cells. Solving the corresponding matrix equation, the handoff rate calculation for all the cells and all the classes is seen to have a complexity $O\left(\delta^{2.376}M\right)$.

To calculate the offered traffic to every cell given the handoff rates has a complexity of $O(7NM)$, since at most every cell is adjacent to six other cells which gives six additions and one more addition for the new call arrival, and we have N cells in the network. Now, let I be the number of iterations needed to obtain convergence in the repeated substitution for the solution of the fixed point. Then it follows that the complexity of calculating the stationary distribution of the states for all the cells is $O\left(I\left(S^{2.376}N + \delta^{2.376}M + 7NM\right)\right)$, which is dominated by $O\left(I\left(S^{2.376}N\right)\right)$.

For the shadow price algorithm, we need to solve the partial derivatives in (7.19) which are obtained each as the solution of a system of S simultaneous equations for each cell i and each class. By solving the corresponding matrix equation, we have that the solution to equation (7.19), for the entire network, has a complexity of $O(S^{2.376}2MN)$. The set of δM linear simultaneous equations in (7.28) needs to be solved. By solving the corresponding matrix equation, it can be seen that the solution of this set of simultaneous equations has complexity $O((\delta M)^{2.376})$. Thus the algorithm for the evaluation of shadow prices has complexity $O\left(I\left(S^{2.376}N\right) + (\delta M)^{2.376} + S^{2.376}2MN\right)$, which is dominated by $O\left(S^{2.376}N\left(I + 2M\right)\right)$.

For the 10 cell network with 15 channels in every cell and two traffic classes with one and two channels as bandwidth requirement for each class, we have: $N = 10$,

$M = 2$, $S = 72$, $b_1 = 1$, $b_2 = 2$, $C = 15$ and $\delta = 38$, which results in a complexity of $O(258,834\ (I+4))$.

7.4 Applications of Shadow Prices

This section contains applications of the shadow price methodology applied to multirate wireless networks. One application is for subscriber pricing in the context of non-uniform traffic. Another is the calculation of Sum Revenue and its use for the comparison of different strategies for new calls and handoff calls.

7.4.1 Pricing

The utility of the shadow prices for pricing policy seems obvious. At an operating point with established traffic levels, knowing the shadow prices of the network net revenue with respect to each of the new call arrival rates can allow the network manager to provision penalties/discounts in order to improve the marginal rewards of additional traffic to the network. Tables 7.2, 7.3 and 7.4 list the shadow prices at an operating point for the 10-cell network in the single rate case and $T = 0, 1$ on all the cells for two cases of mobility, namely, low and high mobility (recall that for the low mobility the departure probabilities for new calls and handoff calls are 0.85 and 0.95, respectively, and for the high mobility case these probabilities are 0.35 and 0.6). The tables show the revenues, w_i, and costs, c_i, values used for the high and low mobility in the calculation of the shadow price.

In Table 7.4, the cost of dropping a handoff was increased considerably compared to that of tables 7.2 and 7.3. Table 7.2 shows the shadow price of a 10-cell *symmetric* network, so called because the capacity is the same for all the cells.

Table 7.2: Pricing Application for a Ten-Cell Symmetric Network

		Shadow Price			
		Low Mobility		High Mobility	
Cell i	λ_i	$T=0$	$T=1$	$T=0$	$T=1$
1	7.50	0.8963	0.8499	-0.2509	0.0151
2	7.50	0.9815	0.9291	0.0552	0.2820
3	7.50	0.8963	0.8499	-0.2509	0.0151
4	7.50	0.9815	0.9291	0.0552	0.2820
5	7.50	1.0104	0.9576	0.2667	0.4540
6	7.50	1.0038	0.9504	0.2464	0.4343
7	7.50	1.0104	0.9576	0.2667	0.4540
8	7.50	1.0104	0.9576	0.2667	0.4540
9	7.50	1.0038	0.9504	0.2464	0.4343
10	7.50	1.0104	0.9576	0.2667	0.4540
Low Mobility $w_i = 1.1579$, $c_i = 2.1579$, High Mobility $w_i = 2.0833$, $c_i = 3.0833$					

In this case the capacity was chosen to be of 15 which is the average capacity of the network in Figure 7.3 and the new call arrival rate λ_i was chosen to be 7.5 for every call i. From the table, it can be seen that the highest shadow price is for those cells with the minimum number of neighbors which is three and with the minimum offered traffic, which in this case are cells 5, 7, 8 and 10. Cells 6 and 9 have also three neighbors, but the offered traffic to them from their neighbors is more than that of the cells mentioned previously. The smallest shadow price is for cells 1 and 3 that have the most number of neighbors. In all the cases analyzed, low and high mobility with and without reservation, the cells with the highest shadow price have the least number of neighbors and those with the lowest shadow price have the most number of neighbors. In the low mobility the shadow price decreases in all the cells when the reservation is increased. In contrast, the shadow price increases in the high mobility case, improving performance by giving priority to handoff calls that are blocked if no reservation is used.

Table 7.3: Pricing Application for a Ten-Cell Network

		Shadow Price			
		Low Mobility		High Mobility	
Cell i	λ_i	$T=0$	$T=1$	$T=0$	$T=1$
1	6.00	0.7507	0.7084	-0.3236	-0.0684
2	7.00	0.9660	0.9102	0.0727	0.2955
3	9.00	0.9802	0.9422	-0.1439	0.1193
4	11.00	1.0749	1.0518	0.3517	0.5291
5	10.00	1.0676	1.0410	0.4844	0.6248
6	5.50	0.8785	0.8059	0.0042	0.2074
7	6.50	0.9696	0.9078	0.2178	0.3998
8	5.50	0.8978	0.8208	0.0407	0.2520
9	8.50	1.0413	1.0001	0.3626	0.5442
10	6.00	0.9150	0.8454	0.0279	0.2353
Low Mobility $w_i = 1.1579$, $c_i = 2.1579$, High Mobility $w_i = 2.0833$, $c_i = 3.0833$					

From Table 7.3, it can be seen that to improve the marginal rewards to the network, traffic in cell 4 (the cell with highest shadow price) should be encouraged more than in cell 1 (the cell with smallest shadow price) for the low mobility case, whereas in the high mobility case, traffic in cell 5 (the highest shadow price) should be encouraged more. This occurs because cell 4 has four neighbors and therefore its offered traffic increases when the mobility increases, raising the blocking and decreasing the shadow price. It can also be seen that for the high mobility case for all the cells the shadow price increases when the reservation increases. This is intuitive since in the high mobility case there are a considerable number of handoffs and protecting them through reservation increases the shadow price. For the cells with the most neighbors, cell 1 and 3, the shadow price is negative in the high mobility case with no reservation and by reserving one channel in every cell the shadow price of cell 3 becomes positive hence improving performance. The shadow price of cell 1 in that case remains negative even though it has the same number of neighbors as cell 3 does, but the offered traffic to it is higher than that of cell 3.

From Table 7.4, it can be seen that to improve the marginal rewards to the network, traffic in cell 4 should be encouraged more than in cell 1 for the low mobility case with no reservation. It can also be seen that for the high mobility case for all the cells the shadow price becomes negative since the cost of dropping a handoff was increased and the number of handoffs is increased by the mobility. Thus, this is the case where efforts ought to be made to discourage traffic in the cells with the negative shadow prices.

It can be seen that in the low mobility case in Table 7.3 the shadow prices decrease when the reservation increases. In contrast, for the case of Table 7.4 they increase which means that by giving protection to handoff calls when the cost of

Table 7.4: Pricing Application for a Ten-Cell Network

		Shadow Price			
		Low Mobility		High Mobility	
Cell i	λ_i	$T=0$	$T=1$	$T=0$	$T=1$
1	6.00	0.1877	0.5905	-3.7589	-1.5863
2	7.00	0.7766	0.8799	-3.0599	-1.1309
3	9.00	0.7695	0.9055	-3.5779	-1.4318
4	11.00	0.9755	1.0296	-2.5580	-0.9880
5	10.00	0.9613	1.0158	-2.2852	-0.9328
6	5.50	0.5987	0.7626	-3.0692	-1.1351
7	6.50	0.7973	0.8785	-2.7555	-1.0483
8	5.50	0.6625	0.7909	-2.8943	-0.9096
9	8.50	0.9340	0.9836	-2.4075	-0.7605
10	6.00	0.6864	0.8147	-2.9246	-0.9562
Low Mobility $w_i = 1.1579$, $c_i = 11.1579$, High Mobility $w_i = 2.0833$, $c_i = 12.0833$					

dropping one is high, the improvement in the net revenue is increased. From the examples, the utility of shadow prices to a service provider in formulating call pricing is obvious.

7.4.2 Sum Revenue

Shadow prices capture the effect of increases in new call external traffic in one cell on the *entire* network. As a result, they are useful in optimizing network-wide goals. Define the *sum revenue* as the *maximum* sum of new call arrival rates such that the new call blocking probability of each cell is less than or equal to some prespecified maximum blocking probability. The notion of sum revenue is similar to that of sum capacity for circuit switched networks. Sum capacity was introduced and calculated for adaptive routing schemes in [70] and [67], where shadow prices were used to solve a nonlinear constrained optimization algorithm. As an illustration of the use of shadow prices in optimizing network-wide goals, we use them to calculate the sum revenue. To this end, we formulate a constrained nonlinear optimization problem with the objective function being the network net revenue and constraints being the new call blocking and handoff drop probabilities. The independent variables are the new call arrival rates. Let $\underline{\eta}$ and $\underline{\gamma}$ be vectors whose components represent the maximum new call and handoff blocking probabilities, respectively, for each cell and let $\underline{0}$ be the zero vector. Then the optimization problem is:

$$\begin{aligned}
\max_{\underline{\lambda}} \quad W(\mathbf{B}, \mathbf{B}_h, \underline{\lambda}, \underline{\alpha}) &= \sum_{m=1}^{M} \sum_{i \in \mathcal{N}} w_{i,m} \lambda_{i,m} \left(1 - B_{i,m}(\underline{\lambda}, \mathbf{p})\right) \\
&\quad - \sum_{m=1}^{M} \sum_{i \in \mathcal{N}} c_{i,m} B_{hi,m}(\underline{\lambda}, \mathbf{p}) \left\{ \mathcal{I}_{\{T_m > 0\}} \alpha_{i,m}(\mathbf{v}) \right. \\
&\quad \left. + \mathcal{I}_{\{T_m = 0\}} \left[\rho_{i,m}(\underline{\lambda}, \mathbf{v}) - \lambda_{i,m} \right] \right\}, \\
\text{subject to} \quad & \mathbf{B} \leq \underline{\eta}, \quad \mathbf{B}_h \leq \underline{\gamma}, \quad \underline{\lambda} \geq \underline{0}.
\end{aligned} \tag{7.29}$$

The solution for the above optimization problem gives the maximum revenue that the network can generate for a given blocking probability vector. The optimization is achieved by using the shadow prices in a gradient descent algorithm that gives the direction in which the vector of new call arrival rates has to be varied to get the desired maximization. The specific algorithm used was the variable metric method [62] using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) update formula.

The step size is obtained by doing a line minimization of Powell's penalty function [62] and the algorithm is stopped when the improvement in the objective function $W(\mathbf{B}, \mathbf{B}_h, \underline{\lambda}, \underline{\alpha})$ is less than 10^{-7}, each handoff blocking probability, $B_{hi,m}$ is in the interval, $(\gamma_{i,m} - 10^{-4}, \gamma_{i,m})$ and each new call blocking probability, $B_{i,m}$ is in the interval, $(\eta_{i,m} - 10^{-4}, \eta_{i,m})$ and the change in each of the decision variables $\lambda_{j,m}$, $\forall j \in \mathcal{N}$, $m = 1, 2, \ldots, M$ is no greater than 10^{-4}.

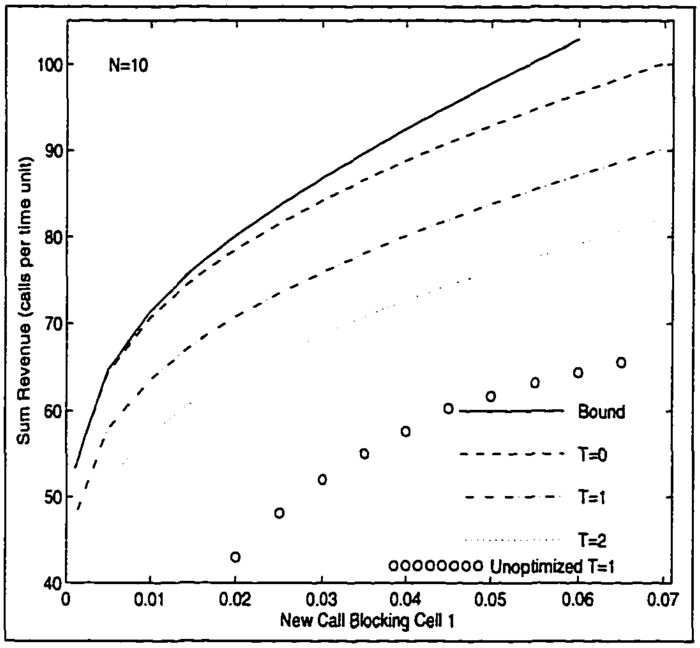

Figure 7.13: Sum Revenue for the 10-Cell Network

It is not known whether the objective function in the above optimization is concave and so it may be possible for the gradient descent algorithm to converge to local minima. In order to ensure that this did not occur we verified the result of the optimization using simulated annealing by starting at one of the solutions obtained by our algorithm and perturbing the decision variables [1]. In all our examples there were no improvements. Sum revenue results for the 10-cell network using different channel reservation parameters for the single rate case can be found in Figure 7.13, where channel reservation strategies for handoffs are compared. The maximum allowed handoff blocking probability was considered in the constraints as 25% of the corresponding new call blocking. The figure includes the Max-Flow Bound of [18], and it can be noted that for the examples conducted with no channel reservation, i.e., $T = 0$, the sum revenue performs close to the bound. The figure also includes the unoptimized case for $T = 1$, i.e., an example where the new call arrivals chosen are different from those obtained by the solution of the optimization problem. The use of shadow prices to maximize revenue results in a significant improvement as it can be seen by comparing the unoptimized case with the optimized case, providing evidence that matching capacity distribution to traffic is important.

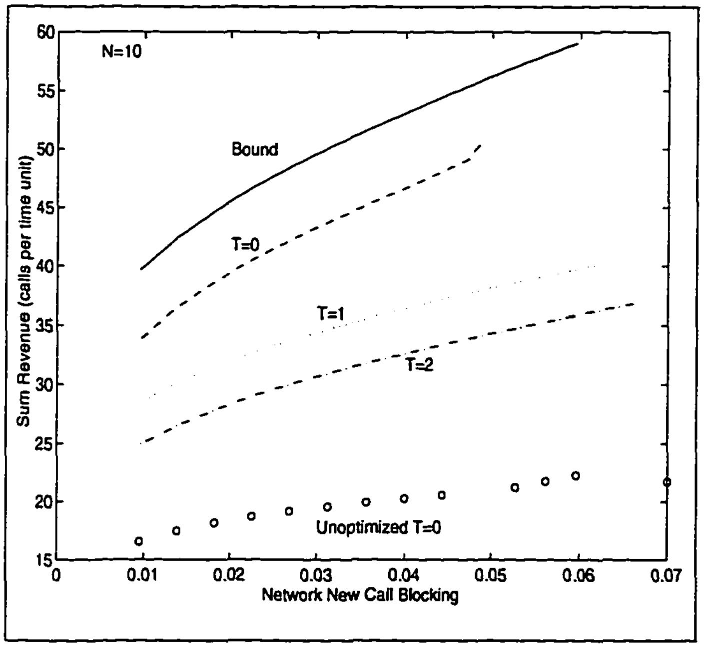

Figure 7.14: Sum Revenue for 10-Cell Network, High Mobility

Figure 7.14 contains the sum revenue for the 10-cell single rate network for the high mobility case for $T = 0, 1, 2$ on every cell. The revenues, w_i, were set at 2.0833, and the costs, c_i, at 3.0833. The figure shows the bound and an unoptimized case with no reservation. The unoptimized case is an example where the new call arrivals chosen are different from those obtained by the solution of the optimization problem. From the figure, it can be seen that increasing mobility decreases sum revenue since more handoff calls will be rejected incurring costs in the net revenue, W and because more new calls will be blocked since the duration of a call will increase.

In Figure 7.15 the sum revenue of the 10-cell network with two classes of customers is shown for several values of channel reservation parameters. The horizontal axis is the new call blocking of class 2 since this is the class with higher bandwidth requirement and its new call blocking is higher than the new call blocking probability of class 1 and the handoff drop probability of both classes. It can be seen that the best performance was obtained for the case of $T = [2, 0]$, where there are two channels reserved for handoffs of class 1. The poorest performance was from the case of two channels reserved for both classes and for the cases $T = [0, 1]$, $T = [1, 1]$ and $T = [2, 1]$, there is no significant difference among their performance in terms of the sum revenue.

The second best case was $T = [1, 0]$ with one channel reserved for class 1, which is in agreement with the best case of $T = [2, 0]$ and it was followed by the case of no channel reserved for any class. This figure also includes the unoptimized case for $T = [2, 0]$ which is the case with highest sum revenue in this example. It can be seen that the use of shadow prices to maximize the sum revenue result in a significant improvement in the sum revenue. Moreover, it can be seen that the unoptimized case performs better than some optimized cases with different reservation parameters, indicating the need for careful choice of reservation parameters for each class of traffic in the multirate case.

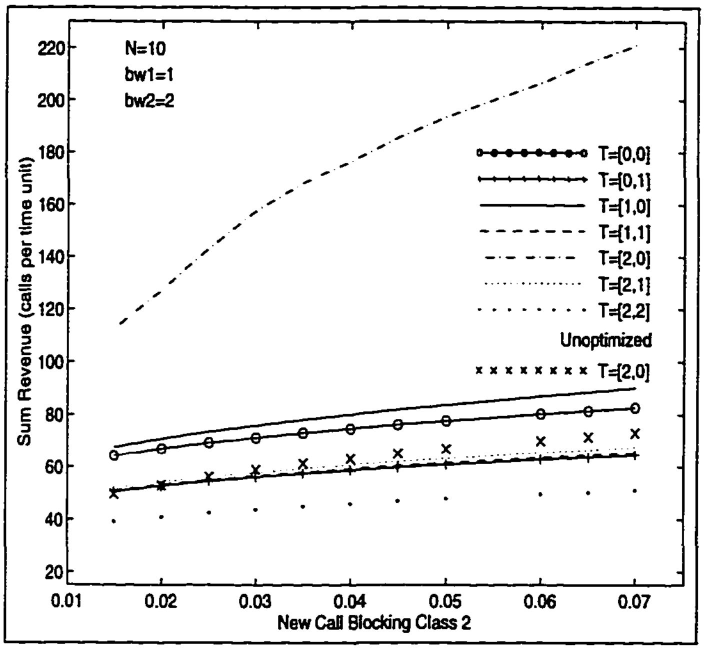

Figure 7.15: Sum Revenue for 10-Cell Network, $\mu = [1, \frac{1}{2}]$

From Figure 7.15, it can be concluded that increasing the channel reservation for the class with less bandwidth requirement improves the sum revenue, whereas increasing it for the other class will degrade the performance.

The calculation of the shadow prices with respect to the capacity and reservation is carried out in the following sections. The model use is for a single rate wireless network, i.e., $M = 1$ in the model of Section 7.2. In order to calculate the shadow prices, we need to know the functional forms used of the performance measures (new call blocking and handoff drop probability) and the distribution of the states, therefore in the next section we describe the model for the single rate case with its fixed point algorithm and later the shadow price with respect to capacity calculation.

7.5 Model for Single-Rate Wireless Networks

Consider an asymmetric cellular network with fixed channel assignment where $\mathcal{N}$ is the set of cells and N the total number of cells. Each cell i has $C_{i,1}$ channels assigned to it. Let $\mathcal{A}_i$ be the set of cells adjacent to cell i. The new call arrival process to cell i is a Poisson process with mean λ_i independent of other new call arrival processes. The time a call remains in cell i, the dwell time, is a random variable with exponential distribution and mean $1/\mu_i$ and it is independent of earlier arrival times, call durations and elapsed times of other users. At the end of a dwell time a call may attempt a handoff to an adjacent cell or leave the network. In this model a *new* call in progress in any cell is treated differently from a *handoff* call for the purposes of termination. We have two *levels* of probabilities for each cell: Let $q_{ij}^{(1)}$ be the probability that a *new* call in progress in cell i after completing its dwell time *goes* to cell j, i.e., there is a first handoff from cell i to cell j, let $q_{ij}^{(2)}$ be the probability that a *handoff* call in progress in cell i after completing its dwell time *goes* to cell j. If cell i and cell j are not adjacent then $q_{ij}^{(s)} = 0$ for $s = 1, 2$. Let $q_{ii}^{(1)}$ be the probability of a departure from the network from cell i when the call in progress is a new call and $q_{ii}^{(2)}$ be the probability of a departure from the network from cell i when the call in progress is a handoff call.

Let the state of a cell be the number of calls present in that cell. Define the set of *feasible* states of cell i as $\Omega_i = \{m : 0 \le m \le C_{i,1}\}$. All the cells have a channel reservation parameter $C_{i,2} > 0$ with $C_{i,2} \le C_{i,1}$, and define $T_i = C_{i,1} - C_{i,2}$ as the number of channels reserved on cell i. The reservation parameter is intended to give priority to handoff calls over new calls through a reservation policy as follows. Define $\mathcal{U}_i$ as the set of unreserved states for cell i, i.e., $\mathcal{U}_i = \{m \in \Omega_i : m < C_{i,2} - 1\}$ and $\mathcal{B}_i$ as the set of reserved states of cell i, i.e., $\mathcal{B}_i = \{m \in \Omega_i : C_{i,2} \leq m < C_{i,1}\}$. When a new call arrives to cell i, it is accomodated if the occupancy of cell i is in the unreserved states, i.e., in $\mathcal{U}_i$, otherwise, it is blocked. If a handoff call arrives to cell i, it is blocked only if all the capacity of cell i is occupied.

We consider that occupancy of the cells evolves according to a birth-death process independent of other cells, where the arrival rate or offered traffic to cell i is ρ_i for the unreserved states and α_i for the reserved states, and the departure rate when cell i is in state $n \in \Omega_i$ is equal to the number of calls connected on that cell times the dwell time, i.e., $n\mu_i$. The birth-death process that describes the occupancy levels of any cell in the network is shown in Figure 7.16, where the subindex i has been dropped.

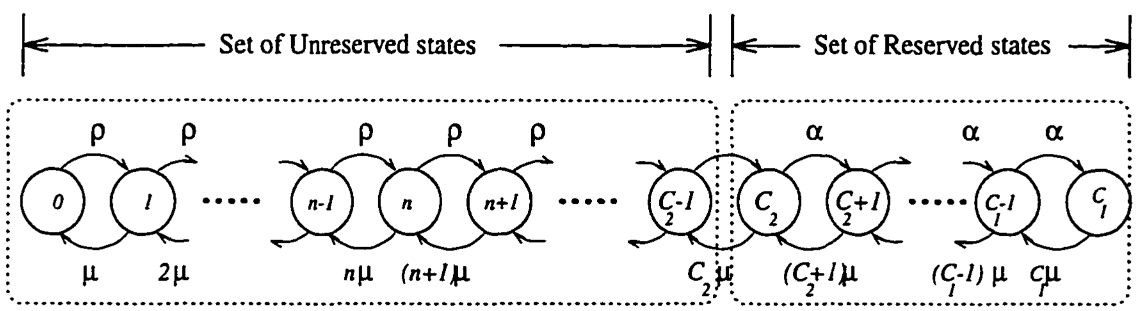

Figure 7.16: Birth-Death Process for One Cell with Capacity C_1

Let $p_i(n)$ be the stationary probability that cell i is in state $n \in \Omega_i$, B_{hi} be the handoff drop probability in cell i and B_i be the blocking probability of new calls in cell i. The solution for the distribution of the states for cell i can be obtained from the following equations from the birth-death process

$$p_i(n) = \left(\frac{\rho_i}{\mu_i}\right)^n \frac{p_i(0)}{n!}, \quad 0 < n \le C_{i,2}, \tag{7.30}$$

$$p_i(n) = \left(\frac{\alpha_i}{\mu_i}\right)^{n-C_{i,2}} \left(\frac{\rho_i}{\mu_i}\right)^{C_{i,2}} \frac{p_i(0)}{n!}, \quad C_{i,2} < n \le C_{i,1}, \tag{7.31}$$

$$p_i(0) = \left\{1 + \sum_{n=1}^{C_{i,2}} \left(\frac{\rho_i}{\mu_i}\right)^n \frac{1}{n!} + \sum_{n=C_{i,2}+1}^{C_{i,1}} \left(\frac{\alpha_i}{\mu_i}\right)^{n-C_{i,2}} \left(\frac{\rho_i}{\mu_i}\right)^{C_{i,2}} \frac{1}{n!}\right\}^{-1}. \tag{7.32}$$

Let ν_{ji} be the handoff rate out of cell j offered to cell i, for adjacent cells i and j. The handoff traffic that can be offered from cell j to an adjacent cell i depends on the proportion of new calls accepted in cell j that goes into cell i, i.e., $\lambda_j(1-B_j)q_{ji}^{(1)}$, and the proportion of handoff calls accepted from cells adjacent to cell j that goes into cell i, i.e., $(1 - B_{hj})q_{ji}^{(2)} \sum_{x \in \mathcal{A}_j} \nu_{xj}$. Thus, the handoff rate out of cell j offered to cell i is given by

$$\nu_{ji} = \lambda_j(1 - B_j)q_{ji}^{(1)} + (1 - B_{hj})q_{ji}^{(2)} \sum_{x \in \mathcal{A}_j} \nu_{xj}. \tag{7.33}$$

Figure 7.2 contains a representation of the terms involved in equation (7.33) where only cells adjacent to cell i are shown. This set of linear simultaneous equations in ν_{ji} can be solved to compute the total offered traffic to cell i, which is given by

$$\rho_i = \lambda_i + \sum_{j \in \mathcal{A}_i} \nu_{ji}\ , \quad n \in \mathcal{U}_i, \tag{7.34}$$

$$\alpha_i = \sum_{j \in \mathcal{A}_i} \nu_{ji}\ , \quad n \in \mathcal{B}_i \backslash \{C_{i,1}\}, \tag{7.35}$$

where "\" is the set subtraction operation. The new call blocking probability, B_i, and the handoff drop probability, B_{hi}, in cell i are given by

$$B_i = \sum_{n=C_{i,2}}^{C_{i,1}} p_i(n), \tag{7.36}$$

$$B_{hi} = p_i(C_{i,1}). \tag{7.37}$$

Equations (7.30)- (7.37) are solved by repeated substitutions to find a solution to the fixed point equations.

Table 7.5: Parameter Values of a Ten-Cell Network, ($C_{1,1}$ was varied)

Parameters at Base Demand							
Cell i	$C_{i,1}$	λ_i	μ_i	w_i	c_i	$q_{ii}^{(1)}$	$q_{ii}^{(2)}$
1	*	5.87584	1.0	1.1579	5.00	0.85	0.95
2	14	7.34580	1.0	1.1579	5.00	0.85	0.95
3	18	10.4375	1.0	1.1579	5.00	0.85	0.95
4	22	13.6520	1.0	1.1579	5.00	0.85	0.95
5	20	12.0385	1.0	1.1579	5.00	0.85	0.95
6	11	5.16572	1.0	1.1579	5.00	0.85	0.95
7	13	6.61500	1.0	1.1579	5.00	0.85	0.95
8	11	5.16572	1.0	1.1579	5.00	0.85	0.95
9	17	9.65164	1.0	1.1579	5.00	0.85	0.95
10	12	5.87584	1.0	1.1579	5.00	0.85	0.95

In order to study the effect on blocking of new and handoff calls, a number of numerical examples were conducted. We present examples in the following, corresponding to the network in Figure 7.3, where it is shown the topology of the network, the number of channels assigned per cell, i.e., $C_{i,1}$, the service rates of each cell, μ_i, and the departure probabilities, $q_{ii}^{(1)}$ and $q_{ii}^{(2)}$ which were chosen to be homogeneous for all the cells with 85% for departure of new calls, and 95% for handoff calls. The ten-cell network has an average capacity of 15 in each cell, and its parameters are in Table 7.5. The new call arrival was chosen such that the new call blocking of all the cells in the network is of 1% with no handoffs taking place. This new call arrival rate will be refered as the *base demand.* In the examples conducted, the base demand is considered to be 0% overload (0% ov. in the figures), and a 100% overload is twice as much demand. In the figures presented, $N = x$ refers to a network with x cells and $T = y$ to y number of channels reserved in all the cells.

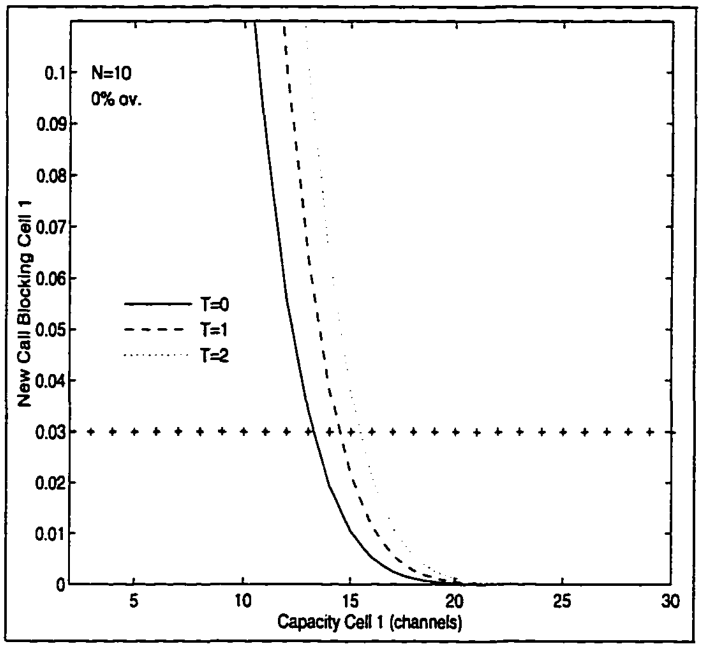

Figure 7.17: New Call Blocking of Cell 1 as its Capacity varies for the Ten-Cell Network

Figure 7.17 contains the new call blocking probability for the ten-cell network as the number of channels in cell 1 was increased for different reservations. The new call arrival was kept constant at the base demand. From Figure 7.17, it can be seen that when the load into the network is the base demand, the number of channels to achieve good performance, around 3 percent blocking, in cell 1 is 13 for the case of no reservation, whereas for one channel reserved in all the cells this number increases to 15 and for two channels reserved a capacity of 16 is needed. Figure 7.18 shows the handoff drop probability for the ten cell network corresponding to the new call

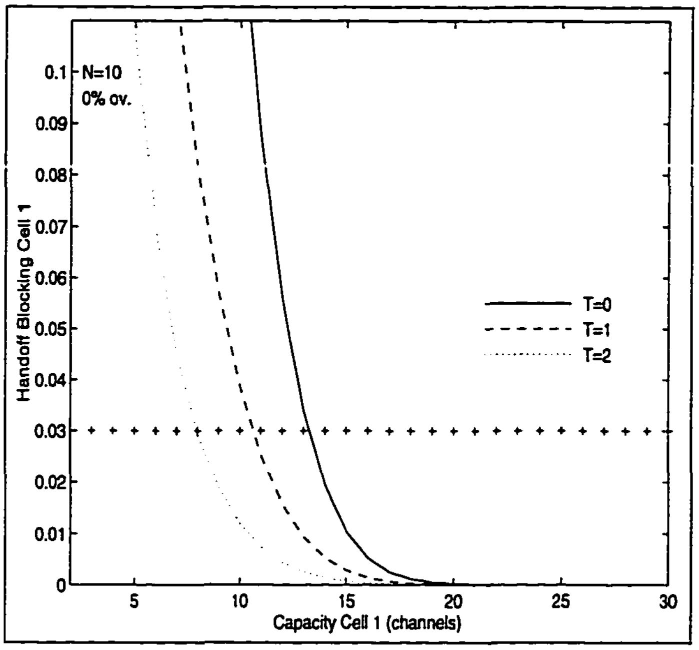

Figure 7.18: Handoff Blocking of Cell 1 as its Capacity varies for the Ten-Cell Network

blocking results of Figure 7.17. It can be seen that as the reservation increases, the handoff drop probability decreases, thus the capacity of cell 1 necessary to achieve the blocking levels desired is decreased.

7.6 Shadow Prices with Respect to Capacity

Following the derivation in Section 7.3, define the net revenue, W, as the revenue generated by the traffic which is carried succesfully. This revenue consists of two components: the first one is the revenue generated by accepting in each cell j a new call, the second takes into account the cost of a forced termination due to handoff failure of those new calls that have arrived and been accepted in cell j, hence the net revenue is

$$\begin{aligned} W(\mathbf{B}, \mathbf{B}_h, \underline{\alpha}, \mathbf{C}) &= \sum_{i\in\mathcal{N}} \Big\{ w_i\lambda_i\left(1 - B_i(\mathbf{C}, \mathbf{p})\right) - c_i B_{hi}(\mathbf{C}, \mathbf{p}) \\ &\quad \cdot \left[\mathcal{I}_{\{C_{i,1} > C_{i,2}\}}\alpha_i(\mathbf{v}) + \mathcal{I}_{\{C_{i,1} = C_{i,2}\}}\left(\rho_i(\mathbf{v}) - \lambda_i\right)\right] \Big\}, \end{aligned} \tag{7.38}$$

where w_j is the revenue generated by accepting a call in cell j, and c_j is the cost of a forced termination of a call due to a handoff failure, $\mathbf{v}$ denotes the vector whose components are the handoff rates ν_{ji} for all adjacent cells i, j, i.e., $\mathbf{v} = (\nu_{12}, \nu_{13}, \ldots, \nu_{ji}, \ldots)$, $\mathbf{p}$ denotes the vector whose components are the stationary probabilities for each state of all the cells, $\mathbf{B}$ is the vector of the new call blocking probabilities for all the cells, $\mathbf{B}_h$ the vector of the handoff blocking probabilities for all the cells, $\mathbf{C}$ the vector whose elements are the capacities and reservation parameters of all the cells and $\underline{\alpha}$ the vector of offered traffics to the cells, let $\mathcal{I}_{\{D\}}$ be 1 if event D is true and 0 otherwise. The second term in (7.38) is the cost due to the proportion of rejected handoff call attempts in cell i.

The choice of the revenue for cell i, w_i, depends on the average holding time of the calls. The average holding time depends on the average number of handoffs the calls undergo before departing from the network and since every time a call is accepted in a cell its duration in that cell is the dwell time with mean $1/\mu_i$ for cell i, the average holding time of a call will be given by the average number of handoffs times the dwell time. Therefore, we take the revenues, w_i, as the average holding time.

Figure 7.19 shows the net revenue for the 10-cell network as the number of channels of cell 1 increases for different reservations. It can be seen that the net revenue becomes almost constant after increasing the number of channels sufficiently to get low blocking levels, i.e., blocking is near zero. It can also be seen that when the number of channels is more than that mentioned to obtain 3% blocking, the case

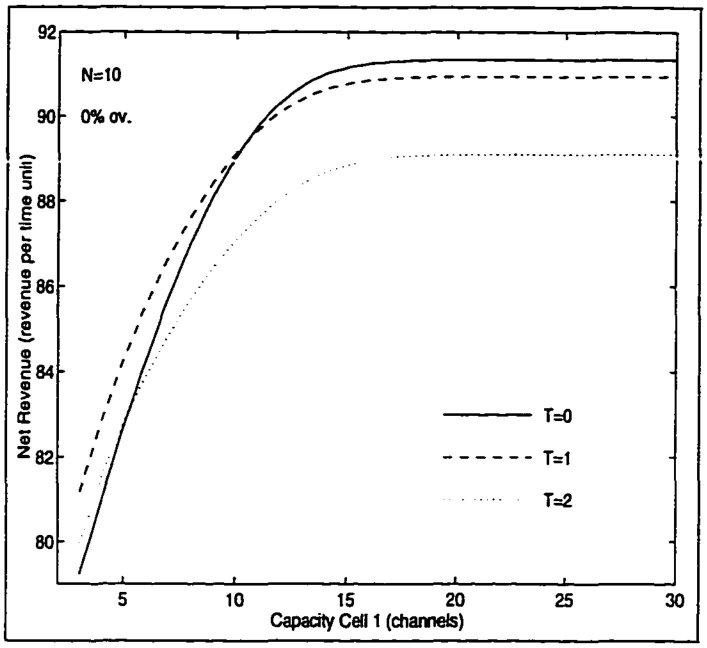

Figure 7.19: Net Revenue for the Ten-Cell Network as Capacity of Cell 1 varies

of no reservation, i.e., $T = 0$, has a higher value of net revenue than the reservation cases, but when the number of channels in this cell is less than 10, the case of one channel reserved, i.e., $T = 1$, performs better than the no reservation case, this is because when the capacity is scarce, more handoffs will be dropped incurring costs, and these costs degrade the net revenue more when no reservation is considered.

We also analyzed for the 10-cell network the cases of low and high mobility, where for low mobility the departure probabilities for new calls, $q_{ii}^{(1)}$, and handoff calls, $q_{ii}^{(2)}$, are 0.85 and 0.95, respectively, and for high mobility these probabilities are 0.35 and 0.6. To analyze the effect of mobility and overload, we considered the new call arrival for each cell at the base demand. We increased the demand from 0% to 110% overload and calculated the net revenue for the 10-cell network.

We varied the capacity of cell 1 keeping constant the capacities of the other cells. The calculation was done for $T = 0, 1$ for all the cells. The revenues, w_i, for low

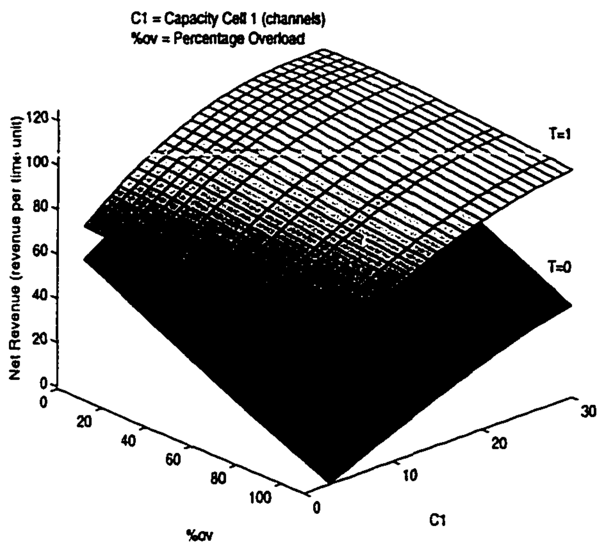

Figure 7.20: Net Revenue as Capacity and Overload Increase (Ten-Cell Network, High Mobility)

mobility were 1.1579 and for high mobility were 2.0833, and the costs, c_i, were set at 5.0 for both cases. Figure 7.19 shows the net revenue for the low mobility case where the case of no reservation, $T = 0$, performs better than the reservation $T = 1, 2$, for capacities greater than 10 channels. Figure 7.20 shows a three dimensional surface of the net revenue, W, versus overload and capacity for cell 1 for the high mobility case. It can be seen that for the high mobility case, the use of reservation improves performance over the case of no reservation due to the increase on the number of handoffs in the network that are given priority by reserving one channel and that would be rejected otherwise incurring costs. Figure 7.21 shows a three dimensional surface of the net revenue, W, versus overload and reservation for cell 1 for the high mobility case when the reservation in the remaining cells is kept constant at $T = 0$ and $T = 1$. It can be seen that with high mobility, the net revenue for the case of $T = 1$ is better than the case of no reservation because of the costs incurred by dropping handoffs due to the high mobility of the customers. The figure shows that for high traffic, the network revenue improves significantly with use of reservation to reduce handoff drop.

The fixed point model describes the $\mathbf{p}$ as an implicit function of $\mathbf{C}$. $\mathbf{B}$ and $\mathbf{B}_h$ are, in turn, functions of $\mathbf{p}$ and thereby an implicit function of $\mathbf{C}$. Consequently, $W(\mathbf{B}, \mathbf{B}_h, \underline{\alpha}, \mathbf{C})$ is also an implicit function of $\mathbf{C}$. We therefore undertake a careful and extensive effort to obtain relations of total and partial derivatives of the new call and handoff blocking probabilities by differentiating the fixed point equations. These relations are manipulated to obtain a system of linear equations in the derivatives of the new call and handoff blocking probabilities with respect to capacities and reservation parameters. Since the capacities are given in terms of number of channels assigned, i.e., an integer number, the derivative will be considered to be defined for all the integer values of capacities as the left derivative, where applicable. Consider that for noninteger values of the capacities linear interpolation is used.

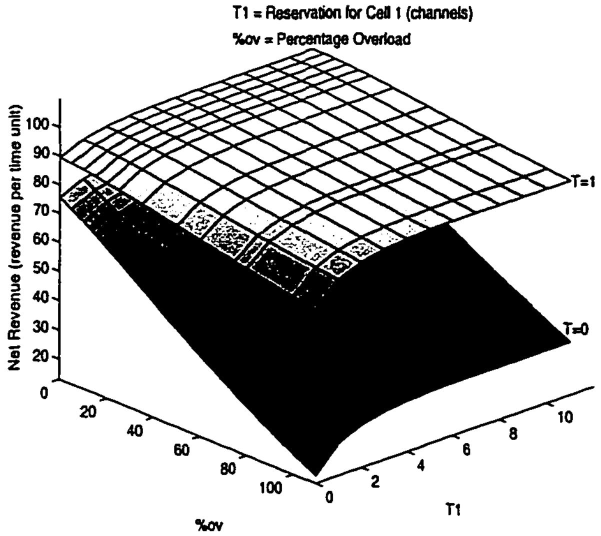

Figure 7.21: Net Revenue as Reservation and Overload Increase (Ten-Cell Network, High Mobility)

7.6.1 Calculation of the Shadow Price

We first need to define the total derivative of the net revenue function with respect to the capacity and reservation parameter as follows

$$\begin{aligned}\frac{dW(\mathbf{B},\mathbf{B}_h,\underline{\alpha},\mathbf{C})}{dC_{k,s}} &= \sum_{i\in\mathcal{N}}\left\{\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\alpha},\mathbf{C})}{\partial B_i}\frac{dB_i(\mathbf{C},\mathbf{p})}{dC_{k,s}}\right.\\ &+\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\alpha},\mathbf{C})}{\partial B_{hi}}\frac{dB_{hi}(\mathbf{C},\mathbf{p})}{dC_{k,s}}\\ &+\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\alpha},\mathbf{C})}{\partial \alpha_i}\frac{d\alpha_i(\mathbf{v})}{dC_{k,s}}\\ &\left.+\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\alpha},\mathbf{C})}{\partial \rho_i}\frac{d\rho_i(\mathbf{v})}{dC_{k,s}}\right\},\end{aligned} \tag{7.39}$$

where $s = 1,2$, $B_i(\mathbf{C},\mathbf{p})$ and $B_{hi}(\mathbf{C},\mathbf{p})$ are given by equations (7.36) and (7.37) and are written here to explicitly show their dependence on $\mathbf{C}$ and $\mathbf{p}$. The partial derivatives needed in the shadow price calculations are the following

$$\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\alpha},\mathbf{C})}{\partial B_i} = -w_i\lambda_i, \tag{7.40}$$

$$\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\alpha},\mathbf{C})}{\partial B_{hi}} = -c_i\left\{\mathcal{I}_{\{C_{i,1}>C_{i,2}\}}\alpha_i(\mathbf{v})+\mathcal{I}_{\{C_{i,1}=C_{i,2}\}}\left[\rho_i(\mathbf{v})-\lambda_i\right]\right\}, \tag{7.41}$$

$$\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\alpha},\mathbf{C})}{\partial \alpha_i} = -c_iB_{hi}(\mathbf{C},\mathbf{p})\mathcal{I}_{\{C_{i,1}>C_{i,2}\}}, \tag{7.42}$$

$$\frac{\partial W(\mathbf{B},\mathbf{B}_h,\underline{\alpha},\mathbf{C})}{\partial \rho_i} = -c_iB_{hi}(\mathbf{C},\mathbf{p})\mathcal{I}_{\{C_{i,1}=C_{i,2}\}}. \tag{7.43}$$

The rest of the terms in (7.39) are obtained as follows. From (7.36) and (7.37) we get

$$\frac{dB_{hi}(\mathbf{C},\mathbf{p})}{dC_{k,s}} = \frac{\partial B_{hi}(\mathbf{C},\mathbf{p})}{\partial C_{k,s}}+\frac{dp_i(\underline{\alpha}_i,C_{i,1})}{dC_{k,s}}, \tag{7.44}$$

$$\frac{dB_i(\mathbf{C},\mathbf{p})}{dC_{k,s}} = \frac{\partial B_i(\mathbf{C},\mathbf{p})}{\partial C_{k,s}}+\sum_{n=C_{i,2}}^{C_{i,1}}\frac{dp_i(\underline{\alpha}_i,n)}{dC_{k,s}}, \tag{7.45}$$

where $\underline{\alpha}_i$ is the vector of the offered traffic to cell i in state given by (7.34) and (7.35). Define $B_i(\mathbf{C},\mathbf{p},C_{i,1}-1)$ as the new call blocking probability when cell i has

$C_{i,1}-1$ channels, obtained by (7.36). Similarly, define $B_{hi}(\mathbf{C},\mathbf{p},C_{i,1}-1)$ as the handoff drop probability when cell i has $C_{i,1}-1$ channels, obtained by (7.37). The first term in (7.45) is given by

$$\frac{\partial B_i(\mathbf{C},\mathbf{p})}{\partial C_{k,s}} = \mathcal{I}_{\{i=k\}}\left[B_i(\mathbf{C},\mathbf{p},C_{k,s}) - B_i(\mathbf{C},\mathbf{p},C_{k,s}-1)\right]. \tag{7.46}$$

From the handoff blocking probability in (7.37) we obtain

$$\frac{\partial B_{hi}(\mathbf{C},\mathbf{p})}{\partial C_{k,s}} = \mathcal{I}_{\{i=k\}}\left[B_{hi}(\mathbf{C},\mathbf{p},C_{k,s}) - B_{hi}(\mathbf{C},\mathbf{p},C_{k,s}-1)\right]. \tag{7.47}$$

Now, the total derivative needed in (7.44) and (7.45) can be obtained from (7.30) - (7.32) as follows

$$\frac{dp_i(\underline{\alpha}_i,n)}{dC_{k,s}} = \frac{\partial p_i(\underline{\alpha}_i,n)}{\partial \rho_i}\frac{d\rho_i(\mathbf{v})}{dC_{k,s}} + \mathcal{I}_{\{C_{i,1}>C_{i,2}\}}\frac{\partial p_i(\underline{\alpha}_i,n)}{\partial \alpha_i}\frac{d\alpha_i(\mathbf{v})}{dC_{k,s}}, \tag{7.48}$$

where $\rho_i(\mathbf{v})$ and $\alpha_i(\mathbf{v})$ are the offered traffic to cell i in the unreserved and reserved states, respectively, given by (7.34) and (7.35) and they are written here to explicitly show their dependence on the handoff rates $\mathbf{v}$.

Equation (7.48) needs the partial derivative of the distribution of the states of cell i with respect to the offered traffic to this cell. Fix n as the state of cell i, if $n \in \mathcal{U}_i$, and $n > 0$, from (7.30) we have the partial derivative with respect to the offered traffic in the unreserved states as

$$\begin{aligned}\frac{\partial p_i(\underline{\alpha}_i,n)}{\partial \rho_i} &= \frac{n}{\mu_i(n!)}\left(\frac{\rho_i(\mathbf{v})}{\mu_i}\right)^{n-1} p_i(\underline{\alpha}_i,0) + \left(\frac{\rho_i(\mathbf{v})}{\mu_i}\right)^{n}\frac{1}{n!}\frac{\partial p_i(\underline{\alpha}_i,0)}{\partial \rho_i} \\ &= \frac{n}{\rho_i(\mathbf{v})}p_i(\underline{\alpha}_i,n) + \frac{p_i(\underline{\alpha}_i,n)}{p_i(\underline{\alpha}_i,0)}\frac{\partial p_i(\underline{\alpha}_i,0)}{\partial \rho_i},\end{aligned} \tag{7.49}$$

when n is in the set of reserved states, i.e., $n \in \mathcal{B}_i$, from (7.31) we get

$$\frac{\partial p_i(\underline{\alpha}_i,n)}{\partial \rho_i} = \frac{C_{i,2}}{\mu_i(n!)}\left(\frac{\rho_i(\mathbf{v})}{\mu_i}\right)^{C_{i,2}-1}\left(\frac{\alpha_i(\mathbf{v})}{\mu_i}\right)^{n-C_{i,2}} p_i(\underline{\alpha}_i,0)$$

$$
\begin{aligned}
& +\left(\frac{\rho_i(\mathbf{v})}{\mu_i}\right)^{C_{i,2}}\left(\frac{\alpha_i(\mathbf{v})}{\mu_i}\right)^{n-C_{i,2}}\frac{1}{n!}\frac{\partial p_i(\underline{\alpha}_i,0)}{\partial \rho_i} \\
= & \frac{C_{i,2}}{\rho_i(\mathbf{v})}p_i(\underline{\alpha}_i,n)+\frac{p_i(\underline{\alpha}_i,n)}{p_i(\underline{\alpha}_i,0)}\frac{\partial p_i(\underline{\alpha}_i,0)}{\partial \rho_i}.
\end{aligned} \tag{7.50}
$$

Now, for the partial derivative of the distribution of the states with respect to the offered traffic in the reserved states when $n \in \mathcal{U}_i$, from (7.30) we have

$$
\frac{\partial p_i(\underline{\alpha}_i,n)}{\partial \alpha_i}=\left(\frac{\rho_i(\mathbf{v})}{\mu_i}\right)^{n}\frac{1}{n!}\frac{\partial p_i(\underline{\alpha}_i,0)}{\partial \alpha_i}=\frac{p_i(\underline{\alpha}_i,n)}{p_i(\underline{\alpha}_i,0)}\frac{\partial p_i(\underline{\alpha}_i,0)}{\partial \alpha_i}, \tag{7.51}
$$

and for $n \in \mathcal{B}_i$, from (7.31) we have

$$
\begin{aligned}
\frac{\partial p_i(\underline{\alpha}_i,n)}{\partial \alpha_i} = & \frac{n-C_{i,2}}{\mu_i(n!)}\left(\frac{\rho_i(\mathbf{v})}{\mu_i}\right)^{C_{i,2}}\left(\frac{\alpha_i(\mathbf{v})}{\mu_i}\right)^{n-C_{i,2}-1}p_i(\underline{\alpha}_i,0) \\
& +\left(\frac{\rho_i(\mathbf{v})}{\mu_i}\right)^{C_{i,2}}\left(\frac{\alpha_i(\mathbf{v})}{\mu_i}\right)^{n-C_{i,2}}\frac{1}{n!}\frac{\partial p_i(\underline{\alpha}_i,0)}{\partial \alpha_i} \\
= & \frac{n-C_{i,2}}{\alpha_i(\mathbf{v})}p_i(\underline{\alpha}_i,n)+\frac{p_i(\underline{\alpha}_i,n)}{p_i(\underline{\alpha}_i,0)}\frac{\partial p_i(\underline{\alpha}_i,0)}{\partial \alpha_i}.
\end{aligned} \tag{7.52}
$$

All the equations above require the partial derivative of $p_i(\underline{\alpha}_i,0)$ with respect to the offered traffic in the reserved, $\alpha_i(\mathbf{v})$, and unreserved, $\rho_i(\mathbf{v})$, states. This is obtained from (7.32) as follows

$$
\begin{aligned}
\frac{\partial p_i(\underline{\alpha}_i,0)}{\partial \alpha_i} = & -p_i(\underline{\alpha}_i,0)^2\cdot\left\{\sum_{n=C_{i,2}+1}^{C_{i,1}}\left(\frac{\alpha_i(\mathbf{v})}{\mu_i}\right)^{n-C_{i,2}-1}\frac{n-C_{i,2}}{\mu_i(n!)}\left(\frac{\rho_i(\mathbf{v})}{\mu_i}\right)^{C_{i,2}}\right\} \\
= & -p_i(\underline{\alpha}_i,0)\left\{\sum_{n=C_{i,2}+1}^{C_{i,1}}\frac{n-C_{i,2}}{\alpha_i(\mathbf{v})}p_i(\underline{\alpha}_i,n)\right\},
\end{aligned} \tag{7.53}
$$

$$
\begin{aligned}
\frac{\partial p_i(\underline{\alpha}_i,0)}{\partial \rho_i} = & -p_i(\underline{\alpha}_i,0)^2\left\{\sum_{n=1}^{C_{i,2}}\frac{n}{\mu_i(n!)}\left(\frac{\rho_i(\mathbf{v})}{\mu_i}\right)^{n-1}\right. \\
& \left.+\frac{C_{i,2}}{\mu_i(n!)}\left(\frac{\rho_i(\mathbf{v})}{\mu_i}\right)^{C_{i,2}-1}\sum_{n=C_{i,2}+1}^{C_{i,1}}\left(\frac{\alpha_i(\mathbf{v})}{\mu_i}\right)^{n-C_{i,2}}\right\} \\
= & -\frac{p_i(\underline{\alpha}_i,0)}{\rho_i(\mathbf{v})}\left\{\sum_{n=1}^{C_{i,2}}np_i(\underline{\alpha}_i,n)+(C_{i,2})\sum_{n=C_{i,2}+1}^{C_i}p_i(\underline{\alpha}_i,n)\right\},
\end{aligned} \tag{7.54}
$$

Equation (7.48) requires the total derivative of the offered traffic with respect to $C_{k,s}$, this can be obtained as follows

$$\frac{d\rho_i(\mathbf{v})}{dC_{k,s}} = \sum_{x\in\mathcal{A}_i} \frac{\partial\rho_i(\mathbf{v})}{\partial\nu_{xi}} \frac{d\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{dC_{k,s}}, \tag{7.55}$$

$$\frac{d\alpha_i(\mathbf{v})}{dC_{k,s}} = \sum_{x\in\mathcal{A}_i} \frac{\partial\alpha_i(\mathbf{v})}{\partial\nu_{xi}} \frac{d\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{dC_{k,s}}, \tag{7.56}$$

To continue with the calculation of the shadow prices, we need the partial derivatives of the offered traffic with respect to the handoff rates, ν_{xi}, which are obtained in the following. From (7.34) or (7.35) we obtain $\frac{\partial\rho_i(\mathbf{v})}{\partial\nu_{xi}} = \mathcal{I}_{\{x\in\mathcal{A}_i\}}$. It can be seen that $\frac{\partial\rho_i(\mathbf{v})}{\partial\nu_{xi}} = \frac{\partial\alpha_i(\mathbf{v})}{\partial\nu_{xi}}$.

Finally, from (7.33), the derivatives of the handoff rates with respect to the parameters $C_{k,s}$ are obtained as

$$\frac{d\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{dC_{k,s}} = \sum_{y\in\mathcal{N}} \left\{ \frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_y} \frac{dB_y(\mathbf{C},\mathbf{p})}{dC_{k,s}} + \frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy}} \frac{dB_{hy}(\mathbf{C},\mathbf{p})}{dC_{k,s}} \right\}, \tag{7.57}$$

where the partial derivatives needed in (7.57) are obtained from the set of simultaneous equations in (7.33) as follows:

$$\frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_y} = -\lambda_x q_{xi}^{(1)} \mathcal{I}_{\{x=y\}} + (1 - B_{hx}(\mathbf{C},\mathbf{p}))\, q_{xi}^{(2)} \sum_{z\in\mathcal{A}_x} \frac{\partial\nu_{zx}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_y}. \tag{7.58}$$

We can see that (7.58) is also a set of simultaneous equations in $\frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_y}$ to be solved. Similarly, for the partial derivative of $\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})$ with respect to $B_{hx}(\mathbf{C},\mathbf{p})$ from (7.33) we can obtain the following

$$\begin{aligned} \frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy}} &= (1 - B_{hx}(\mathbf{C},\mathbf{p}))\, q_{xi}^{(2)} \sum_{z\in\mathcal{A}_x} \frac{\partial\nu_{zx}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy}} \\ &\quad -\mathcal{I}_{\{x=y\}} q_{xi}^{(2)} \sum_{z\in\mathcal{A}_x} \nu_{zx}(\mathbf{B},\mathbf{B}_h,\mathbf{v}), \end{aligned} \tag{7.59}$$

which is also a set of simultaneous equations in $\frac{\partial \nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy}}$ to be solved. Substituting (7.55) and (7.56) into (7.48) together with (7.49) - (7.54), and these into (7.44) and (7.45), we can bring them to (7.57) to obtain a system of linear simultaneous equations in $\frac{d\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{dC_{k,s}}$ as follows

$$\begin{aligned}
\frac{d\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{dC_{k,s}} = & \sum_{y\in\mathcal{N}}\Bigg[\frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_y}\frac{\partial B_y(\mathbf{C},\mathbf{p})}{\partial C_{k,s}} \\
& +\frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy}}\frac{\partial B_{hy}(\mathbf{C},\mathbf{p})}{\partial C_{k,s}}\Bigg] \\
& +\sum_{y\in\mathcal{N}}\sum_{z\in A_y}\Bigg\{\sum_{n=C_{y,2}}^{C_{y,1}}\Bigg(\frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_y}\frac{\partial p_y(\underline{\alpha}_y,n)}{\partial\rho_y}\frac{\partial\rho_y(\mathbf{v})}{\partial\nu_{zy}} \\
& +\mathcal{I}_{\{C_{y,1}>C_{y,2}\}}\frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_y}\frac{\partial p_y(\underline{\alpha}_y,n)}{\partial\alpha_y}\frac{\partial\alpha_y(\mathbf{v})}{\partial\nu_{zy}}\Bigg) \\
& +\frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy}}\frac{\partial p_y(\underline{\alpha}_y,C_y)}{\partial\rho_y}\frac{\partial\rho_y(\mathbf{v})}{\partial\nu_{zy}} \\
& +\mathcal{I}_{\{C_{y,1}>C_{y,2}\}}\frac{\partial\nu_{xi}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{\partial B_{hy}}\frac{\partial p_y(\underline{\alpha}_y,C_y)}{\partial\alpha_y}\frac{\partial\alpha_y(\mathbf{v})}{\partial\nu_{zy}}\Bigg\}\frac{d\nu_{zy}(\mathbf{B},\mathbf{B}_h,\mathbf{v})}{dC_{k,s}},
\end{aligned} \tag{7.60}$$

which is solved and the result used in (7.55), (7.56), (7.48), (7.44) and (7.45) to substitute into (7.39).

Figure 7.22 shows the shadow price with respect to the capacity of cell 1 of the net revenue. It can be seen that the shadow price with respect to capacity decreases meaning that the blocking decreases and the net revenue becomes constant.

7.7 Model Simplifications

In the previous section, the shadow price methodology was presented and the derivatives with respect to the capacities and the reservation parameters were obtained. In this section we discuss the reduction of the methodology presented to some particular cases.

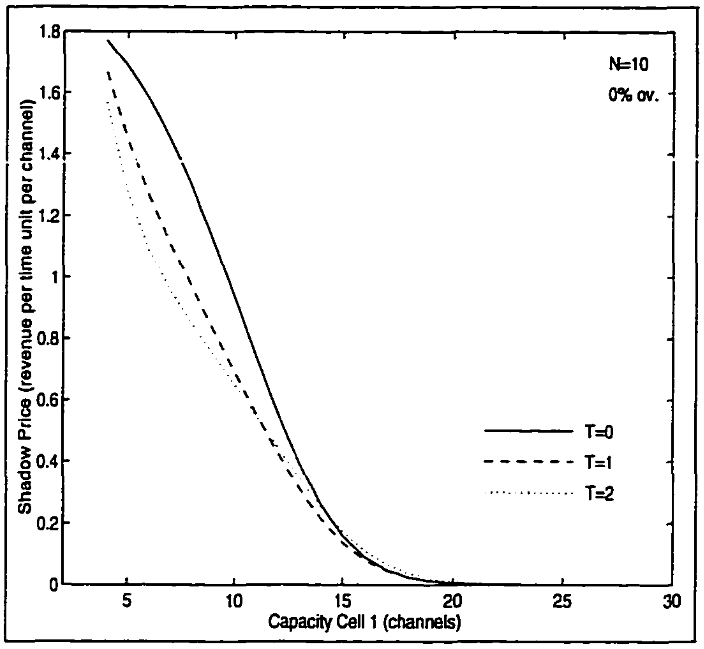

Figure 7.22: Shadow Price with Respect to Capacity of Cell 1 for the Ten-Cell Network

7.7.1 No Handoffs and No Channel Reservation

The simplest model is when there are no handoffs and no channel reservation. This case can be obtained by setting $C_{i,1} = C_{i,2}$, $\alpha_i = 0$ and $\rho_i = \lambda_i$ for each cell i in the network, hence the net revenue, W, becomes the network rate of return

$$W(\mathbf{B}, \underline{\lambda}, \mathbf{C}) = \sum_{i \in \mathcal{N}} w_i \lambda_i \left(1 - B_i(\mathbf{C}, \mathbf{p})\right). \tag{7.61}$$

This case is reduced to a similar shadow price calculation as that in [35]. It can be seen that the new call blocking of cell i will be given by the Erlang B function, i.e., $B_i = E(\frac{\lambda_i}{\mu_i}, C_{i,1})$. Now, since $\frac{dB_i}{d\lambda_k} = (1 - B_k)\,\eta_k$, where $\eta_k = E(\frac{\lambda_k}{\mu_k}, C_{k,1} - 1) - E(\frac{\lambda_k}{\mu_k}, C_{k,1})$ is the probability of a channel being seized, and using $\frac{\partial W(\mathbf{B},\underline{\lambda},\mathbf{C})}{\partial \lambda_k} = w_k\,(1 - B_k)$ and equation (7.40), we can obtain the shadow price with respect to the new call arrival λ_k as follows

$$\frac{dW(\mathbf{B}, \underline{\lambda}, \mathbf{C})}{d\lambda_k} = w_k\,(1 - B_k) - w_k \lambda_k \eta_k\,(1 - B_k), \tag{7.62}$$

Now, since $\frac{dB_i}{dC_k} = -\eta_k$ and considering (7.40), we get that the shadow price with respect to capacity is given by $\frac{dW(\mathbf{B},\underline{\lambda},\mathbf{C})}{dC_{k,1}} = w_k \lambda_k \eta_k$. Hence, we can conclude that

$$\frac{dW(\mathbf{B},\underline{\lambda},\mathbf{C})}{d\lambda_k} = (1 - B_k)\left[w_k - \frac{dW(\mathbf{B},\underline{\lambda},\mathbf{C})}{dC_{k,1}}\right], \tag{7.63}$$

which shows that for each new call that is accepted on cell k, a revenue of w_k is generated at a cost of $\frac{dW(\mathbf{B},\underline{\lambda},\mathbf{C})}{dC_{k,1}}$ for using the channel on that cell, hence $\frac{dW(\mathbf{B},\underline{\lambda},\mathbf{C})}{dC_{k,1}}$ can be interpreted as the *cost* of using one channel on cell k and $w_k - \frac{dW(\mathbf{B},\underline{\lambda},\mathbf{C})}{dC_{k,1}}$ as the net revenue generated by carrying the call on cell k.

7.7.2 No Channel Reservation and One Level of Mobility

This simplification considers that the departure of handoff calls and new calls are treated equally. This reduces the offered traffic (7.34) to

$$\rho_i = \lambda_i + \sum_{j \in \mathcal{A}_i} \mu_j E[N_j] q_{ji}, \tag{7.64}$$

where N_j is the random variable for the number of calls in progress for cell j and $E[N_j] = \sum_{n=0}^{C_{j,1}} n p_j(n)$ is the expected value of N_j, which has distribution $p_j(n)$ given by (7.30), (7.31) and (7.32). This expectation has the following expression

$$E[N_j] = \frac{\lambda_j}{\mu_j}(1 - B_j) + \frac{\sum_{k \in \mathcal{A}_j} \nu_{kj}}{\mu_j}(1 - B_{hj}), \tag{7.65}$$

and since we are considering the case of no channel reservation, we have $B_j = B_{hj}$ which reduces (7.65) to

$$E[N_j] = \frac{(1 - B_j)}{\mu_j}\left[\lambda_j + \sum_{k \in \mathcal{A}_j} \nu_{kj}\right]. \tag{7.66}$$

On the other hand, the handoff rate into cell i from cell j given by (7.33) becomes

$$\nu_{ji} = q_{ji}(1 - B_j)\left[\lambda_j + \sum_{k \in \mathcal{A}_j} \nu_{kj}\right], \tag{7.67}$$

and the total departure rate from cell j which is given by $\mu_j E[N_j]$, is the sum of the handoff rates out of cell j and the departures out of the network from cell j. Hence, the handoff rate out of cell j offered to cell i, which is given in (7.67), can be expressed as $\nu_{ji} = q_{ji}\mu_j E[N_j]$, which substituted in (7.34) gives the desired result (7.64). Since the blocking probability can be expressed in terms of the Erlang B function, i.e., $B_i = B_{hi} = E(\frac{\rho_i}{\mu_i}, C_{i,1})$, substituting (7.66) into (7.64) and considering equation (7.34), we obtain

$$\rho_i = \lambda_i + \sum_{j \in \mathcal{A}_i} q_{ji}\mu_j \left(\frac{\rho_j}{\mu_j}\right)\left(1 - E\left(\frac{\rho_j}{\mu_j}, C_{j,1}\right)\right) = \lambda_i + \sum_{j \in \mathcal{A}_i} q_{ji}\rho_j\,(1 - B_j)\,, \tag{7.68}$$

which in turn gives, together with the Erlang B function, the fixed point equations. The net revenue is given by

$$W(\mathbf{B}, \underline{\alpha}, \mathbf{C}) = \sum_{i \in \mathcal{N}} \left\{ w_i\lambda_i\,(1 - B_i(\mathbf{C}, \mathbf{p})) - c_i B_i(\mathbf{C}, \mathbf{p})\,[\rho_i(\underline{\lambda}, \mathbf{C}, \mathbf{p}) - \lambda_i] \right\}. \tag{7.69}$$

then the following result is obtained

Theorem 7.1 Assume the conditions of the fixed point model of Section 7.5 with the simplifications of no channel reservation and one level of mobility as explained above, then the shadow prices with respect to the new call arrival and with respect to the capacities are related to each other in the following form

$$\begin{aligned}\frac{dW(\mathbf{B}, \underline{\alpha}, \mathbf{C})}{d\lambda_k} &= (1 - B_k)\left(w_k - \frac{dW(\mathbf{B}, \underline{\alpha}, \mathbf{C})}{dC_{k,1}}\frac{1}{\mu_k}\right) \\ &\quad + \sum_{\substack{i \in \mathcal{N}, \\ i \neq k}} \frac{dW(\mathbf{B}, \underline{\alpha}, \mathbf{C})}{dC_{i,1}}\frac{1}{\mu_i}(1 - B_i)\end{aligned} \tag{7.70}$$

Remark: We can see with respect to (7.63) that the handoffs that take place in the network produce one more term in the expression for the shadow price. The term reflects the cost of the channels potentially used on each cell by the call, conditioned on its acceptance on the cell to which it is handing over.

Proof: From (7.69) we can obtain the following equation

$$\frac{dW(\mathbf{B},\underline{\alpha},\mathbf{C})}{d\lambda_k} = \frac{\partial W(\mathbf{B},\underline{\alpha},\mathbf{C})}{\partial\lambda_k} + \sum_{i\in\mathcal{N}}\left\{\frac{\partial W(\mathbf{B},\underline{\alpha},\mathbf{C})}{\partial B_i}\frac{dB_i(\mathbf{C},\mathbf{p})}{d\lambda_k} + \frac{\partial W(\mathbf{B},\underline{\alpha},\mathbf{C})}{\partial\rho_i}\frac{d\rho_i(\mathbf{v})}{d\lambda_k}\right\}. \tag{7.71}$$

Since the new call blocking is given by the Erlang B formula, we can obtain

$$\frac{dB_i(\mathbf{C},\mathbf{p})}{d\lambda_k} = \mathcal{I}_{\{k\in\mathcal{A}_i\}}\left(1-E\left(\frac{\rho_i}{\mu_i},C_{i,1}\right)\right)\eta_i\frac{1}{\mu_i}\frac{d\rho_i(\mathbf{v})}{d\lambda_k}, \tag{7.72}$$

where $\eta_i = E(\frac{\rho_i}{\mu_i},C_{i,1}-1) - E(\frac{\rho_i}{\mu_i},C_{i,1})$. The derivative of the offered traffic with respect to the new call arrival is

$$\frac{d\rho_i(\mathbf{v})}{d\lambda_k} = \frac{\partial\rho_i(\mathbf{v})}{\partial\lambda_k} + \sum_{x\in\mathcal{A}_i}\frac{\partial\rho_i(\mathbf{v})}{\partial B_x}\frac{dB_x(\mathbf{C},\mathbf{p})}{d\lambda_k}. \tag{7.73}$$

Now, define the following matrices $\beta = \mathrm{diag}(1-B_i)$, $\mu = \mathrm{diag}(\frac{1}{\mu_i})$, $\eta = \mathrm{diag}(\eta_i)$, then substituting (7.73) into (7.72) we obtain

$$\begin{aligned}\frac{dB_i(\mathbf{C},\mathbf{p})}{d\lambda_k} &= \mathcal{I}_{\{k\in\mathcal{A}_i\}}\left(1-E\left(\frac{\rho_i}{\mu_i},C_{i,1}\right)\right)\eta_i\frac{1}{\mu_i}\frac{\partial\rho_i(\mathbf{v})}{\partial\lambda_k}\\ &+\mathcal{I}_{\{k\in\mathcal{A}_i\}}\left(1-E\left(\frac{\rho_i}{\mu_i},C_{i,1}\right)\right)\eta_i\frac{1}{\mu_i}\sum_{x\in\mathcal{A}_i}\frac{\partial\rho_i(\mathbf{v})}{\partial B_x}\frac{dB_x(\mathbf{C},\mathbf{p})}{d\lambda_k},\end{aligned} \tag{7.74}$$

which can be written in matrix form with the previous definitions as follows $\frac{dB(\mathbf{C},\mathbf{p})}{d\lambda} = \Psi^{-1}\left[\beta\ \eta\ \mu\right]$ where $\frac{dB(\mathbf{C},\mathbf{p})}{d\lambda} = \left[\frac{dB_i(\mathbf{C},\mathbf{p})}{d\lambda_k}\right]_{i,k}$. Therefore, (7.73) in matrix form becomes

$$\frac{d\rho(\mathbf{v})}{d\lambda} = I + \frac{\partial\rho(\mathbf{v})}{\partial B}\cdot\frac{dB(\mathbf{C},\mathbf{p})}{d\lambda}, \tag{7.75}$$

where I is the identity matrix. Now, for the shadow price with respect to capacity we have

$$\frac{dB_i(\mathbf{C},\mathbf{p})}{dC_{k,1}} = -\eta_i + \left(1-E\left(\frac{\rho_i}{\mu_i},C_{i,1}\right)\right)\eta_i\frac{1}{\mu_i}\frac{d\rho_i(\mathbf{v})}{dC_{k,1}}, \tag{7.76}$$

$$\frac{d\rho_i(\mathbf{v})}{dC_{k,1}} = \sum_{x\in A_i} \frac{\partial \rho_i(\mathbf{v})}{\partial B_x}\frac{dB_x(\mathbf{C},\mathbf{p})}{dC_{k,1}}, \tag{7.77}$$

which substituted in (7.76) and in matrix form we get

$$\frac{dB(\mathbf{C},\mathbf{p})}{dC} = -\Psi^{-1}\eta = -\Psi^{-1}\beta\,\eta\,\mu\,\beta^{-1}\mu^{-1} = -\frac{dB(\mathbf{C},\mathbf{p})}{d\lambda}\,\beta^{-1}\mu^{-1}, \tag{7.78}$$

where the second equality comes from the fact that the matrices defined are diagonal and their product commutes. Similarly, we can obtain (7.77) in matrix form using (7.78) as follows

$$\frac{d\rho(\mathbf{v})}{dC} = -\frac{\partial\rho(\mathbf{v})}{\partial B}\cdot\frac{dB(\mathbf{C},\mathbf{p})}{d\lambda}\,\beta^{-1}\mu^{-1}, \tag{7.79}$$

Finally, the shadow price with respect to capacity of the net revenue in matrix form is

$$\frac{dW(\mathbf{B},\underline{\alpha},\mathbf{C})}{dC} = \frac{\partial W(\mathbf{B},\underline{\alpha},\mathbf{C})}{\partial B}\frac{dB(\mathbf{C},\mathbf{p})}{dC} + \frac{\partial W(\mathbf{B},\underline{\alpha},\mathbf{C})}{\partial\rho}\frac{d\rho(\mathbf{v})}{dC}, \tag{7.80}$$

and substituting (7.78) and (7.79) in (7.80) we get

$$\frac{dW(\mathbf{B},\underline{\alpha},\mathbf{C})}{dC} = \left(\frac{\partial W(\mathbf{B},\underline{\alpha},\mathbf{C})}{\partial\lambda} - \frac{dW(\mathbf{B},\underline{\alpha},\mathbf{C})}{d\lambda}\right)\beta^{-1}\,\mu^{-1} + \frac{\partial W(\mathbf{B},\underline{\alpha},\mathbf{C})}{\partial\rho}\,\beta^{-1}\,\mu^{-1}, \tag{7.81}$$

from which the desired result (7.70) is obtained.

7.8 Applications of the Shadow Price Calculation

This section contains applications of the shadow price methodology applied to single rate wireless networks. One application is for dimensioning wireless networks, specifically, the allocation of channels or assignment of frequencies to the cells. Another application is the optimal reservation levels that the cells should have to keep a good grade of service. In the subsequent, we present each of the applications in a separate section.

7.8.1 Optimal Assignment of Frequencies

As an illustration of the use of shadow prices in optimizing network-wide goals, we use them to calculate the optimal number of channels that have to be assigned to a cell such that the net revenue is maximized. The number of channels allocated on a cell affects the quality of service by improving or degrading the performance measures such as new call blocking and handoff drop probabilities.

Consider a network of cells where the input demand is known and a set of channels can be assigned to each cell, how many channels should we assign to each cell such that the quality of service is optimal for that traffic pattern?

Using the net revenue in (7.38) as the performance function, we can formulate a constrained nonlinear optimization problem where the shadow prices with respect to capacity are used to maximize the objective function for a given traffic input. The constraints are the new call blocking and handoff drop probabilities. The independent variables are the cell capacities. Let $\underline{\eta}$ and $\underline{\gamma}$ be the vectors whose components represent the maximum new call blocking and handoff blocking probabilities for each cell, let $\mathbf{C_1}$ be the vector of capacities of all the cells and let $\underline{0}$ be the zero vector. Then the optimization problem is:

$$\begin{aligned}\max_{\mathbf{C_1}} \quad W(\mathbf{B}, \mathbf{B}_h, \underline{\alpha}, \mathbf{C}) &= \sum_{i \in \mathcal{N}} w_i \lambda_i \left(1 - B_i(\mathbf{C}, \mathbf{p})\right) - \sum_{i \in \mathcal{N}} c_i B_{hi}(\mathbf{C}, \mathbf{p}) \cdot \\ &\quad \cdot \left\{ \mathcal{I}_{\{C_{i,1} > C_{i,2}\}} \alpha_i(\mathbf{v}) + \mathcal{I}_{\{C_{i,1} = C_{i,2}\}} \left[\rho_i(\mathbf{v}) - \lambda_i\right] \right\}, \\ \text{subject to} \quad & \mathbf{B} \geq \underline{\eta},\ \mathbf{B}_h \geq \underline{\gamma},\ C_{i,1} \geq C_{i,2} \geq 0,\ \forall i \in \mathcal{N}. \quad (7.82)\end{aligned}$$

The solution to the above optimization problem gives the maximum revenue that the network can generate for a given blocking probability vector. The optimization is achieved by using the shadow prices in a gradient descent algorithm that gives the direction in which the vector of capacities has to be varied to get the desired maximization. The result of the optimization was obtained by fixing the new call blocking probability allowed in each cell and the handoff drop probability at 25% of that value, e.g., for a new call blocking of 1% the handoff drop probability is 0.25%.

We can solve this optimization problem by using standard integer nonlinear programming algorithms as the Interior Penalty Function method [60] where the original optimization problem is transformed to a minimization problem with a function that considers penalties for the contributions of the constraints to ensure the search of the solution in the feasible region, and penalties when some of the decision variables are noninteger. Another method is to consider that the shadow prices presented in Section 7.6 are for integer values of the capacities and extend their definition by linear interpolation to noninteger values of the capacities, apply a gradient descent algorithm and solve the optimization problem. Another method is to consider that the fixed point model is extended by the recurrence relations in [21] for noninteger values of the capacities.

At the end of the optimization problem in the last two methods, the vector of capacities will generally be noninteger, once this is obtained, perform a grid search to determine the integer vector of capacities at which the constraints are not violated We used the method of linear interpolation and verified our results with the other two methods mentioned. The solution obtained by the three methods had no difference in the final results.

Table 7.6: Parameters for Optimal Channel Allocation on a Seven-Cell Network

Parameter Values (Low Mobility)						
Cell i	λ_i	μ_i	w_i	c_i	$q_{ii}^{(1)}$	$q_{ii}^{(2)}$
1	5.0	1.0	1.1579	5.00	0.85	0.95
2	12.0	1.0	1.1579	5.00	0.85	0.95
3	17.0	1.0	1.1579	5.00	0.85	0.95
4	6.0	1.0	1.1579	5.00	0.85	0.95
5	3.5	1.0	1.1579	5.00	0.85	0.95
6	4.2	1.0	1.1579	5.00	0.85	0.95
7	7.1	.01	1.1579	5.00	0.85	0.95

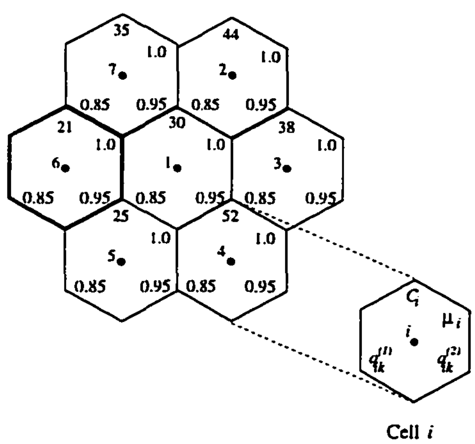

Figure 7.23: Seven-Cell Network Used in Examples with Parameter Values

A series of examples were conducted for a seven-cell network shown in Figure 7.23 and with parameters for the low mobility case shown in Table 7.6. This network forms one of the basic cluster size for channel reuse according to the cellular concept [51]. The high mobility case considered departure probabilities for new calls, $q_{ii}^{(1)}$, of 0.35 and for handoff calls, $q_{ii}^{(2)}$, of 0.6 and revenues, w_i, of 2.0833. Table 7.7

Table 7.7: Optimal Channel Allocation for the Seven-Cell Network

Capacity in Channels for $B_i = 0.03$						
		Low Mobility		High Mobility	
Cell i	λ_i	$T=0$	$T=1$	$T=0$	$T=1$
1	5.0	13	14	28	31
2	12.0	20	21	28	30
3	17.0	25	26	32	34
4	6.0	13	14	20	22
5	3.5	9	10	15	16
6	4.2	10	11	15	17
7	7.1	14	15	20	22

contains the total number of channels per cell after solving the optimization problem formulated. The cases of low and high mobility with no reservation, $T = 0$, and one channel reserved, $T = 1$, are presented for a new call blocking probability, B_i, of 0.03 in all the cells and a handoff drop probability of one fourth this value. It can be seen that in the low mobility case, increasing reservation by one channel results in an increase in capacity of one channel for all the cells, but in the high mobility case the increase in almost all the cells is of more than one channel, and for cell 1 it is higher (three channels) than that of the other cells because it has more neighbors hence more offered calls and in order to keep the blocking levels the capacity has to increase.

Another example was conducted for the seven cell network where the new call arrival for all the cells was kept constant and equal to 7.0 calls per time unit, we will refer to this example as the *symmetric* case. We considered the low and high mobility cases with no reservation and with one channel reserved in all the cells. The results are in Table 7.8, where it can be seen that for cells with the same number of neighbors the capacities are equal, and for the cell with the most neighbors the capacity is higher. It can also be noted that as reservation increases, in the low mobility case, the increase of the number of channels for cell 1 (the cell with the highest number of neighbors) is of one channel whereas the capacities for the other cells increase in two channels. In contrast, for the high mobility case, the increment in the number of channels for cell 1 (three channels) is more than that of the other cells. This is because the number of handoff calls offered to cell 1 increases more than that of the other cells, and to prevent these to be dropped, more channels need to be reserved.

Table 7.8: Optimal Channel Allocation for the Seven-Cell *Symmetric* Network

Capacity in Channels for $B_i = 0.03$					
		Low Mobility		High Mobility	
Cell i	λ_i	$T = 0$	$T = 1$	$T = 0$	$T = 1$
1	7.0	15	16	28	31
2	7.0	13	15	20	21
3	7.0	13	15	20	21
4	7.0	13	15	20	21
5	7.0	13	15	20	21
6	7.0	13	15	20	21
7	7.0	13	15	20	21

7.8.2 Optimal Reservation Levels

Consider a cellular network where the external demand and the capacities of each cell are known. Suppose that we require to have a quality of service measured by the new call blocking and the handoff drop probabilities at a prescribed level. How many channels can we reserve on each cell for handoffs to achieve this goal?

Using the net revenue in (7.38), we can formulate a constrained nonlinear optimization problem as in the previous section where the shadow prices with respect to the channel reservation parameters are used to maximize the objective function. The independent variables are the cell reservation levels $\mathbf{C_2}$. Let $\underline{\eta}$ and $\underline{\gamma}$ be the vectors whose components represent the maximum new call blocking and handoff blocking probabilities for each cell, let $\mathbf{C_2}$ be the vector of reservation parameters for all the cells and let $\underline{0}$ be the zero vector. Then the optimization problem is:

$$\begin{aligned}\max_{\mathbf{C_2}} \quad W(\mathbf{B},\mathbf{B}_h,\underline{\alpha},\mathbf{C}) \quad &= \sum_{i\in\mathcal{N}} w_i\lambda_i\left(1-B_i(\mathbf{C},\mathbf{p})\right) - \sum_{i\in\mathcal{N}} c_i B_{hi}(\mathbf{C},\mathbf{p})\cdot \\ &\quad \cdot\left\{\mathcal{I}_{\{C_{i,1}>C_{i,2}\}}\alpha_i(\mathbf{v}) + \mathcal{I}_{\{C_{i,1}=C_{i,2}\}}\left[\rho_i(\mathbf{v})-\lambda_i\right]\right\}, \\ \text{subject to} \quad &\mathbf{B}\le\underline{\eta},\ \mathbf{B}_h\le\underline{\gamma},\ C_{i,1}\ge C_{i,2}\ge 0,\ \forall i\in\mathcal{N}. \quad (7.83)\end{aligned}$$

Following the same approach as that in Section 7.8.1 to solve the optimization problem, we give an example where these reservation levels are calculated. The example is for the seven cell network of figure 7.23 with parameters shown in Table 7.9. It was considered that the network parameters were as those shown in Table 7.6. The results of the optimization are in Table 7.10, where it can be seen that for the same capacity and new call arrival rates in order to keep a new call blocking of 0.03, the number of channels reserved decreases or remains the same for almost all the cells as mobility increases. This is because mobility will increase handoffs, but protecting them with excessive reservation degrades new call blocking more than what is desired, hence reservation has to be decreased.

Table 7.9: Parameters for Optimal Reservation Levels on a Seven-Cell Network

Parameter Values (Low Mobility)							
Cell i	λ_i	C_i	μ_i	w_i	c_i	$q_{ii}^{(1)}$	$q_{ii}^{(2)}$
1	3.50	14	1.0	1.1579	5.00	0.85	0.95
2	5.25	21	1.0	1.1579	5.00	0.85	0.95
3	6.50	26	1.0	1.1579	5.00	0.85	0.95
4	3.50	14	1.0	1.1579	5.00	0.85	0.95
5	2.50	10	1.0	1.1579	5.00	0.85	0.95
6	2.75	11	1.0	1.1579	5.00	0.85	0.95
7	3.75	15	.01	1.1579	5.00	0.85	0.95

Table 7.10: Optimal Reservation Levels for the Seven-Cell Network

Channels Reserved for $B_i = 0.03$				
			Low Mobility	High Mobility
Cell i	λ_i	$C_{i,1}$	T_i	T_i
1	3.50	14	4	2
2	5.25	21	10	10
3	6.50	26	14	13
4	3.50	14	5	5
5	2.50	10	3	4
6	2.75	11	4	4
7	3.75	15	6	6

Table 7.11: Optimal Reservation Levels for the Seven-Cell *Symmetric* Network

Channels Reserved for $B_i = 0.03$				
			Low Mobility	High Mobility
Cell i	λ_i	$C_{i,1}$	T_i	T_i
1	6.25	25	3	1
2	6.25	25	10	8
3	6.25	25	10	8
4	6.25	25	10	8
5	6.25	25	10	8
6	6.25	25	10	8
7	6.25	25	10	8

A *symmetric* seven cell network was also considered, where the capacities of all the cells were of 25 channels and the new call arrival of 6.25 calls per time unit. The results are in Table 7.11 where it can be seen that reservation is equal for those cells with the same number of channels, and that for the cell with the most neighbors (cell 1), the number of channels reserved is significantly less than that of the other cells. This is because this cell is offered more calls than the others due to the number of neighbors, and reserving more channels in this cell will affect its new call blocking more than the corresponding in the other cells that are offered less calls.

Chapter 8

Conclusions

In this chapter, we present a summary and general conclusions of the research. At the end of the chapter, in Section 8.2, some future research projects are presented which are the result of important questions that we have found doing this work.

8.1 Summary and General Conclusions

This section presents a brief summary of the results and contributions of this research, as well as the conclusions concerning them.

8.1.1 Summary

In Chapter 3, the fixed point algorithms for performance evaluation of LLR [8], ALBA[54], RALBA and MLLR [71] have been presented. Their corresponding routing policies were explained. The networks used in the numerical calculations were introduced, as well as results of the comparison of the routing schemes by Network Blocking Probability.

Bounds used in the numerical results were presented, as well as a comparison of the non-uniqueness of the fixed point [31] model using fixed routing and LLR.

In Chapter 4, we introduced the Shadow Price methodology and calculation for the routing schemes of Chapter 3. This chapter also contains the comparison of the routing schemes by using the network rate of return and the bounds. The Shadow

Prices were verified for LLR by comparing them to simulation results and using 99 % confidence intervals to show that numerical calculation and simulation are in close agreement.

Chapter 5 introduces two applications of the Shadow Prices. The first is the pricing of services where the shadow price gives a criterion to assign penalties/discounts to improve rewards. The other application is for the Sum Capacity where the shadow prices are used in a gradient descent algorithm to maximize the network rate of return in a nonlinear constrained optimization problem formulation.

The Sum Capacity for symmetric and asymmetric networks is presented and bounds on the performance are introduced to determine the closeness to optimal values of some of the examples.

In Chapter 6, the fixed point model for performance evaluation of multirate networks using LLR is presented, as well as the shadow price methodology, simulation results with comparison to analytical results and the sum capacity.

In Chapter 7, an extension of the fixed point model in [53] for performance evaluation of wireless networks is presented, this model considers mobility of the customers at two levels. The first level is for those customers that just arrived into the network, new call arrivals, and characterizes its mobility into adjacent cells or out of the network. The second level is for those customers that have already carry out a handoff into an adjacent cell from where they started to have service. Another modification to the fixed point model of [53] is that we introduce multiple classes of customers into the network.

The shadow price with respect to new call arrival for multirate wireless networks is introduced and the shadow price with respect to capacity for single rate is presented. The shadow price methodology is shown to have several applications on wireless networks, and examples of sum capacity, and most importantly, dimensioning and optimal channel reservation levels are presented and analyzed.

8.1.2 General Conclusions

The shadow price methodology was presented in the context of some applications such as routing in circuit-switched networks and multi-rate networks, optimization of reservation levels, traffic allocation and dimensioning in wireless networks and multirate wireless networks.

It was shown that the shadow price methodology can be applied for pricing of traffic allocation, dimensioning of wireless networks, decision making in routing schemes, maximization of carried traffic, establishment of reservation levels and capacity mismatch. The technique used is based on the application of the chain rule to implicitly defined functions of the external demand and the capacities of the network. The applications were obtained by formulation of optimization problems where the shadow prices are used in a gradient descent algorithm to obtain the decision variables.

We described the calculation of shadow prices of several adaptive routing schemes, LLR [8], ALBA [54], RALBA, MLLR [71] and Multirate LLR, and presented numerical results for the calculation of these shadow prices in several networks. As an application of these shadow prices, we formulated an optimization program which calculates the sum capacity of those routing schemes for a given network. Comparison of the sum capacities indicates that the optimization using shadow prices results in a significant improvement. This provides evidence that matching capacity distribution to traffic is important even when adaptive routing schemes such as LLR and ALBA are used in the network. The numerical results indicate that with just a small number of states the capacity of ALBA approaches that of LLR. The numerical results also illustrate the different effects of asymmetry on the sum capacity behaviour of various adaptive routing schemes, as well as the effect of multiple classes of customers. We also calculated bounds on the sum capacity which indicate the potential amount of loss in sum capacity by using these adaptive routing schemes in circuit-swyched networks.

The shadow price methodology was extended to the case of networks with multiple classes of customers with LLR as routing policy. Numerical results for simulations and analysis were presented. The performance evaluation was shown to depend on the trunk reservation used and on the network load, where for high input traffic and reservation, the numerical and simulation results were in close agreement. The shadow price with respect to external arrival rates were calculated and several cases where the service rate was varied were presented.

Shadow prices were applied on wireless networks were a performance evaluation algorithm that considers mobility of the customers was presented and extended further to the case of multiple classes of traffic. Sum capacity results for these two cases were presented together with a bound as in the case of LLR and ALBA. Channel reservation levels were used to give priority to handoff calls over new call arrivals, and the optimal levels were found by maximizing the net revenue using the shadow prices with respect to the channel reservation parameters in a gradient descent algorithm giving the network characteristics such as topology, demand and capacities as known. Another important network management problem was analyzed, the assignment of channels to the cells with the network topology and demand known. To find such optimal values, the shadow prices with respect to capacities were calculated and used in a nonlinear optimization problem that resulted in the maximization of the net revenue for prescribed values of quality of service. The fixed point algorithm for this application was modified to consider noninteger values of the capacities and the corresponding expressions can be found in Appendix A.

Therefore, shadow prices can be incorporated in the network management and analysis to provide a better understanding of the behavior of telecommunication networks and help their design.

8.2 Future Research

One of the most important questions is how adaptive routing schemes, which follow a performance evaluation algorithm through fixed point equations, would be extended to Asynchronous Transfer Mode (ATM) networks, and how the shadow prices should be considered, because we might be able to obtain a closed form expression for the shadow prices with respect to the link capacities. Another problem found to be analyzed is the application of shadow prices to establish a pricing framework on broadband networks.

The results given by the analysis of MLLR brought up the topic of optimality of LLR, the problem would be to find the characteristics under which LLR would be optimal since for some examples we found that MLLR is better than LLR.

The difficulty to apply the shadow price methodology to online decision making, is based on the complexity of the equations used. Neural network models could be proposed to solve the system of simultaneous linear equations needed to find the shadow price.

It is known that overflow traffic offered to a link due to rejection on another link is not Poisson. In multirate networks this overflow traffic is peaked and it has been modelled by Interrupted Poisson Processes (IPP) in several other works, but no state dependent routing scheme has been considered. It is interesting to know how IPP could be modified such that adaptive routing schemes can be analyzed.

In ATM networks, the traffic models of the sources through the use of Effective Bandwidths have been found by using mean and peak rates, and qualities of service and other criteria. It is important with the increasing use of integrated networks to see the application of effective bandwidths to routing and network management.

Finally, with the advent of new technologies in wireless networks and Personal Communication Systems (PCS) such as Code Division Multiple Access (CDMA), it is important to incorporate the shadow price methodology to problems such as optimal performance in terms of interference levels or probability of error.

I believe that this research has helped to open doors to new possibilities in the analysis of communication networks.

Bibliography

[1] Aarts E. and Korst, J., *Simulated Annealing and Boltzmann Machines: A Stochastic Approach to Combinatorial Optimization and Neural Computing,* John Wiley & Sons, 1989.

[2] Akinpelu, J.M., "The Overload Performance of Engineered Networks with Non-hierarchical and Hierarchical Routing," *A.T. & T. Tech. Jour.*, vol. 63, pp. 1261-1281, 1984.

[3] Asawa, Manjari., "Optimal Scheduling and Admission Control in Communication Networks," *Ph.D. Dissertation*, The University of Michigan, 1995.

[4] Ash, G.R., Cardwell, R.H., and Murray, R.P., "Design and Optimization of Networks with Dynamic Routing," *Bell System Technical Journal*, vol. 60, pp. 1787-1820, 1981.

[5] Ash, G.R., Chen, J.S., Frey, A.E. and Huang B.D., "Real-Time Network Routing in a Dynamic Class-of-Service Network," *Teletraffic and Datatraffic in a Period of Change,* eds. Jensen, A. and Iversen, V.B., North-Holland, 1991.

[6] Ash, G.R. and Huang B.D., "An Analytical Model for Adaptive Routing Networks," *IEEE Trans. on Communications*, vol. 41, No. 11, pp. 1748-1759, November 1993.

[7] Cameron, H., Galloy, P., and Graham, W.J., "Report on the Toronto Advanced Routing Concept Trial," *Network Planning Symposium*, vol. 1, pp. 228-236, 1980.

[8] Chung, S.P., Kaspher, A. and Ross, K.W., "Computing Approximate Blocking Probabilities for Large Loss Networks with State-Dependent Routing." *IEEE/ACM Transactions on Networking*, vol. 1, No. 1, pp. 105-115, February 1993.

[9] Chung, S.P. and Ross, K.W., " Reduced Load Approximations for Multirate Loss Networks," *IEEE Transactions on Communications*, vol. 41, No. 8, pp. 1222-1231, August 1993.

[10] Cox, Donald C., "Wireless Network Access for Personal Communications," *IEEE Communications Magazine,* vol. 30, No. 12, pp. 96-115, December 1992.

[11] Crametz, J.P. and Hunt, P.J., "A Limit Result Respecting Graph Structure for a Fully Connected Loss Network with Alternative Routing," *The Annals of Applied Probability*, vol. 1, No. 3, pp. 436-444, 1991.

[12] Dziong, Z., Pioro, M., Korner, U. and Wickberg, T., "On Adaptive Call Routing Strategies in Circuit Switched Networks - Maximum Revenue Approach," *International Teletraffic Congress 12*, June 1988.

[13] Everitt, D. and Manfield, D., "Performance Analysis of Cellular Mobile Communication Systems with Dynamic Channel Assignment," *IEEE Journal on Selected Areas in Communications*, vol. 7, No. 8, pp. 1172-1179, October 1989.

[14] A. Faragó, S. Blaabjerg, L. Ast, G. Gordos and T. Henk, "A New Degree of Freedom in ATM Network Dimensioning: Optimizing the Logical Configuration," *IEEE Jour. Sel. Areas Comm.*, vol. 13, No. 7, pp. 1199-1206, September 1995.

[15] Fleming, W., *Functions of Several Variables*, Addison-Wesley Publishing Company, 1965.

[16] Gibbens, R.J., Hunt, P.J. and Kelly, F.P., "Bistability in Communication Networks," *Disorder in Physical Systems*, Grimmett, G.R. and Welsh, D.J.A. eds., pp. 113-128, Oxford University Press, 1990.

[17] Gibbens, R.J., Kelly, F.P. and Turner, S.R.E, "Dynamic Routing in Multiparented Networks," *IEEE/ACM Transactions on Networking*, vol. 1, No. 2, pp. 261-270, 1993.

[18] Gibbens, R.J., and Kelly, F.P., "Dynamic routing in fully connected networks," *IMA Journal of Mathematical Control and Information*, vol. 7, pp. 77-111, 1990.

[19] Gibbens R. and Whiting, " An investigation of Implied Cost Methods for Circuit-Switched Network Optimization," *Proceedings, Fifth UK Teletraffic Symposium*, 1988.

[20] Girard, A., *Routing and Dimensioning in Circuit-Switched Networks*, Addison-Wesley Publishing Company, 1990.

[21] Girard, A., "Blocking Probability of Noninteger Groups with Trunk Reservation." *IEEE Transactions on Communications* vol.33, No. 2, pp. 113-120, February 1985.

[22] Girard, A., "Revenue Optimization of Telecommunication Networks." *IEEE Transactions on Communications* vol.41, No. 4, pp. 583-591, April 1993.

[23] Girard, A., "Revenue Optimization of B-ISDN Networks." *IEEE Transactions on Communications* vol.43, No. 5, pp. 1992-1997, May 1995.

[24] Girard, A. and Bouley, H., "Two-Moment Models for the Performance Evaluation of B-ISDN Networks," *Telecommunication Systems*, vol. 3, pp. 61-89, 1994.

[25] Girard, A., and Bell, M.A., "Blocking Evaluation for Networks with Residual-Capacity Adaptive Routing," *IEEE Transactions on Communications*, vol. 37, No. 12, pp. 1372-1380, December 1989.

[26] Girard, A. and Liau, B., "Dimensioning of Adaptively Routed Networks," *IEEE/ACM Trans. on Networking*, vol. 1, No. 4, pp. 460-468, August 1993.

[27] M.V. Hegde, P.S. Min and A. Rayes, "State Dependent Routing: Traffic Dynamics and Performance Benefits," *Journal of Network nd Systems Management,*, vol. 2, No.2, pp.125-148, 1994.

[28] Hong, D., and Rappaport S., "Traffic Model and Performance Analysis for Cellular Mobile Radio Telephone Systems with Prioritized and Nonprioritized Handoff Procedures," *IEEE Trans. on Vehicular Technology*, vol. VT-35, No. 3, August 1986.

[29] Hunt, P.J., "Implied Costs in Loss Networks," *Adv. Appl. Prob.,* vol. 21, pp. 661-680, 1989.

[30] Hunt, P.J. and Laws, C.N., "Asymptotically Optimal Loss Network Control," *Mathematics of Operations Research*, vol. 18, No. 4, pp. 880-900, November 1993.

[31] Kelly, F.P., "Blocking Probabilities in Large Circuit-Switched Networks," *Advances in Applied Probability*, vol. 18, pp. 473-505, 1986.

[32] Kelly, F.P., *Reversibility and Stochastic Networks*, Wiley, Chichester, 1979.

[33] Kelly, F.P., "Loss Networks," *The Annals of Applied Probability*, vol. 1, No. 3, pp. 319-378, 1991.

[34] Kelly, F.P., "Bounds on the Performance of Dynamic Routing Schemes for Highly Connected Networks," *Mathematics of Operations Research*, vol. 19, No. 1, pp. 1-20, February 1994.

[35] Kelly, F.P., "Routing in Circuit-Switched Networks: Optimization, Shadow Prices and Decentralization," *Adv. in App. Prob.*, vol. 20, pp. 112-144, 1988.

[36] Kelly, F.P., "Routing and Capacity Allocation in Networks with Trunk Reservation." *Mathematics of Operations Research*, vol.15, pp. 771-792, Nov 1990.

[37] Key, P.B. and Cope G.A., "Distributed Dynamic Routing Schemes." *IEEE Communications Magazine*, vol. 28, No. 10, pp. 54-64, October 1990.

[38] Key, P.B. and Whitehead M.J. "Cost-Effective Use of Networks Employing Dynamic Alternative Routing," *International Teletraffic Congress 12*, pp.987-997, North-Holland,1989.

[39] Krishnan, K.R. and T.J. Ott, "Forward -Looking Routing: A New State-dependent Routing Scheme," *International Teletraffic Congress 12*, paper 3.4A4, Torino 1988.

[40] Krishnan, K.R., and Ott, T.J., "State-Dependent Routing for Telephone Traffic: Theory and Results," *25th IEEE Control and Decision Conference*, pp. 2124-2128, Athens, December 1986.

[41] Krishnan, K.R., "Load Allocation for Non-Alternate Routing of Telephone Traffic with Possible Application to State Dependent Routing," *Bell Communications Research*, Technical Report TM-ARH-000-488, November 26, 1984.

[42] Krishnan, K.R., "Markov Decision Algorithms for Dynamic Routing," *IEEE Communications Magazine*, vol. 28, No. 10, pp. 66-69, October 1990.

[43] Krishnan, K.R. and Ott, T., "Separable Routing: A Scheme for State-Dependent Routing of Circuit Switched Telephone Traffic," *Annals of Operations Research*, vol. 35, pp. 43-68, 1992.

[44] Krishnan, K.R., " Performance Evaluation of Networks under State-Dependent Routing," *Bellcore Symposium on Performance Modelling,* Livingston, N.J., May 1990.

[45] Kolarov, A. and Hui, J., "Least Cost Routing in Multiple-Service Networks," *IEEE INFOCOM '94*, pp. 1482-89, 1994.

[46] Kolarov, A. and Hui, J., "Least Cost Routing in Multi-Service Networks: Part II," *IEEE INFOCOM '95*, pp. 289-96, 1995.

[47] Lakshmivarahan S., and Dhall, S.D., *Analysis and Design of Parallel Algorithms: Arithmetic and Matrix Problems,* McGraw-Hill, 1990.

[48] Lam, S. and Reiser, M., "Congestion Control of Store-and-Forward Networks by Input Buffer Limits - An Analysis," *IEEE Trans. Comm.*, vol. 27, No. 1, Jan. 1979.

[49] Liao, K-Q. and Mason, L.G., "Multislot Integrated Services System," *IEEE Trans. on Communications*, vol. 37, No. 3, pp. 211-221, March 1989.

[50] Lin, Y-B., Mohan, S. and Noerpel, A., "Analyzing the Trade-off between Implementation Costs and Performance: PCS Channel Assignment Strategies for Hand-off and Initial Access," *IEEE Personal Communications Magazine*, Third Quarter 1994.

[51] Mac Donald, V.H., "Advanced Mobile Phone Service: The Cellular Concept," *Bell System Technical Journal*, vol. 58, No. 1, pp. 15-41, January 1979.

[52] Malinvaud, E., *Lectures on Microeconomic Theory,* North-Holland, Amsterdam, 1972.

[53] McMillan, David, "Traffic Modelling and Analysis for Cellular Mobile Networks," *Proceedings of the ITC-13*, pp. 627-632, 1991.

[54] Mitra, D., Gibbens, R. J. and Huang B. D., "State-dependent Routing on Symmetric Loss Networks with Trunk reservations - I," *IEEE Transactions on Communications,* vol. 41, No. 2, pp. 400-411, February 1993.

[55] Narendra, K. and Thathachar, M.A.L., *Learning Automata, An Introduction*, Prentice Hall, 1989.

[56] Narendra, K. and Mars, P., "The Use of Learning Algorithms in Telephone Traffic Routing - A Methodology," *Automatica*, vol. 19, pp. 495-502, 1983.

[57] Narendra, K. and Thathachar, M.A.L., "On the Behavior of Learning Automata in a Changing Environment with Application to Telephone Traffic Routing," *IEEE Trans. on Systems, Man and Cybernetics*, SMC-10, pp. 262-269, 1980.

[58] Ott, T.J., and Krishnan, K.R., "State-Dependent Routing of Telephone Traffic and the Use of Separable Routing Schemes," *International Teletraffic Congress*, vol. 11, pp. 867-872, 1985.

[59] Rayes, A. and Min, P.S., "Capacity Expansion in State Dependent Routing Schemes," *Proceedings of the 1994 Conference on Information Sciences and Systems*, Princeton, pp. 237-241, March 1994.

[60] Rao, S. S., *Optimization: Theory and Applications*, John Wiley & Sons, Second Edition, 1984.

[61] Rayes, A. and Min, P.S., "Application of Shadow Price in Capacity Expansion of State Dependent Routing," *Journal of Network and Systems Management,* to appear, March 1996.

[62] Reklaitis, G.V., A. Ravindran and K. M. Ragsdell, *Engineering Optimization Methods and Applications,* Wiley Interscience, pp. 452-455, 1983.

[63] Sidi, M., and Starobinski, D., "New Call Blocking versus Handoff Blocking in Cellular Networks," *IEEE INFOCOM 96*, pp. 35-42, March 1996.

[64] Tajima, J. and Imamura, K., "A Strategy for Flexible Channel Assignment in Mobile Communication Systems," *IEEE Transactions on Vehicular Technology*, vol. 37, No. 2, pp. 92-103, May 1988.

[65] Tekinay, S. and Jabbari, B., "Handover and Channel Assignment in Mobile Cellular Networks," *IEEE Communications Magazine*, November 1991.

[66] Vargas, C., Hegde, M. and Naraghi-Pour, M., "Shadow Prices for State Dependent Routing," *The Twenty-Eighth Annual Conference on Information Sciences and Systems*, Princeton University, Princeton, N.J. 08544, March 1994.

[67] Vargas, C., Hegde, M. and Naraghi-Pour, M., "Shadow Prices for LLR and ALBA," accepted for publication to *IEEE/ACM Trans. on Networking*, June 1996.

[68] Vargas, C., Hegde, M. and Naraghi-Pour, M., "Shadow Prices for Wireless Networks," submitted for publication to *IEEE/ACM Trans. on Networking*, March 1996.

[69] Vargas, C., Hegde, M. and Naraghi-Pour, M., "Use of Shadow Prices in Wireless Networks," submitted to *3rd International Telecommunications Symposium*, Acapulco, Mexico, October 1996.

[70] Vargas, C., Hegde, M., Naraghi-Pour, M. and Min, P., "Shadow Prices for State Dependent Routing," *The Twenty-Eighth Annual Conference on Information Sciences and Systems*, Princeton University, Princeton, N.J. 08544, March 1994.

[71] Vargas, C., Hegde, M. and Naraghi-Pour, M., "Shadow Prices for LLR with no Direct Preference (MLLR)," *The 28th IEEE Southeastern Symposium on System Theory*, Louisiana State University, Baton Rouge, Louisiana, 70803. March 31 - April 2, 1996.

[72] Wanamaker, D.M. and Dorrance, D.A., "Dynamically Controlled Routing Field Trial Experience," *Proceedings of the Network Operations and Management Systems Conference*, New Orleans, March 1988.

[73] Wang, W. and Saadawi, T.N., "Trunk Congestion Control in Heterogeneous Circuit Switched Networks," *IEEE Trans. on Communications*, vol. 40, No. 7, pp. 1156-1161, July 1992.

[74] Whitt, W., "Blocking when Service is Required from Several Facilities Simultaneously," *AT&T Technical Journal*, vol. 64, pp. 1807-1856, 1985.

[75] Wong E.W.M and Yum T. S., " Maximum Free Circuit Routing in Circuit-Switched Networks," *Proceeding, INFOCOM '90,* 1990.

[76] Young, E., "Cost Allocation," *Proc. Symp. Appl. Math.,* 33, American Mathematical Society, pp. 69-94, 1985.

[77] Yue, W., "Analytical Methods to Calculate the Performance of a Cellular Mobile Radio Communication System with Hybrid Channel Assignment," *IEEE Transactions on Vehicular Technology*, vol. 40, No. 2, pp. 453-460, May 1991.

[78] Zhang, M., and Yum, T-S., "Comparison of Channel-Assignment Strategies in Cellular Mobile Telephone Systems," *IEEE Transactions on Vehicular Technology*, vol. 38, No. 4, pp. 211-215, November 1989.

[79] Zhang, M., and Yum, T-S., "The Nonuniform Compact Pattern Allocation Algorithm for Cellular Mobile Systems," *IEEE Transactions on Vehicular Technology*, vol. 40, No. 2, pp. 387-391, May 1991.

Appendix A.

A.1 Uniqueness of Offered Traffic

We demonstrate the uniqueness of the solution to equations (7.5), (7.6) and (7.7). Consider first the case of no reservation for all the classes m, i.e., $T_m = 0$, as a result, we have for each class m that $B_{i,m} = B_{hi,m}$. From (7.6) the following can be obtained.

$$\sum_{j \in \mathcal{A}_i} \nu_{ji,m} = \rho_{i,m} - \lambda_{i,m} \ . \tag{A.1}$$

Substituting this in (7.5) gives

$$\nu_{ji,m} = (1 - B_{hj,m}) \left\{ \lambda_{j,m} q^{(1)}_{ji,m} + q^{(2)}_{ji,m} \left[\rho_{j,m} - \lambda_{j,m} \right] \right\}. \tag{A.2}$$

If we substitute equation (A.2) in equation (7.6) we obtain

$$\rho_{i,m} = \lambda_{i,m} + \sum_{j \in \mathcal{A}_i} (1 - B_{hj,m}) \left\{ \lambda_{j,m} q^{(1)}_{ji,m} + q^{(2)}_{ji,m} \left[\rho_{j,m} - \lambda_{j,m} \right] \right\}, \tag{A.3}$$

which can be rewritten as

$$\rho_{i,m} = \lambda_{i,m} + \sum_{j \in \mathcal{A}_i} \lambda_{j,m} (1 - B_{hj,m}) \left[q^{(1)}_{ji,m} - q^{(2)}_{ji,m} \right] + \sum_{j \in \mathcal{A}_i} (1 - B_{hj,m}) q^{(2)}_{ji,m} \rho_{j,m}, \tag{A.4}$$

which is a system of simultaneous equations in $\rho_{i,m}$. Consider the case of one level of mobility, i.e., $q^{(1)}_{ji,m} = q^{(2)}_{ji,m}$, $\forall i, j \in \mathcal{N}$. Then we obtain

$$\rho_{i,m} = \lambda_{i,m} + \sum_{j \in \mathcal{A}_i} (1 - B_{hj,m}) q^{(2)}_{ji,m} \rho_{j,m}, \tag{A.5}$$

in which $\rho_{i,m}$ may be identified with the expression of the offered traffic to a queue in a Jackson network (network of open queues). This has been shown to have a unique solution [32]. Define the following matrices which will be used in the derivations from now on throughout this appendix.

$$R_m^{(1)} = \left[q_{ij,m}^{(1)}\right]_{\substack{i\neq j \\ NxN}}, \; R_m^{(2)} = \left[q_{ij,m}^{(2)}\right]_{\substack{i\neq j \\ NxN}}, \; \beta_m = \text{ diag } \begin{pmatrix} 1 & -B_{hi,m} \end{pmatrix}, \tag{A.6}$$

$$\gamma_m = \text{ diag } (1 - B_{i,m})_{NxN}, \; \mathcal{L}_m = \text{ diag } (\lambda_{i,m})_{NxN}, \; B_{hm} = [B_{hi,m}]_{Nx1}, \tag{A.7}$$

$$\mathcal{P}_m = \text{ col } (\rho_{i,m}, \forall i \in \mathcal{N}), \; \mathbf{c}_m = \text{ row } (c_{i,m}, \forall i \in \mathcal{N}), \; B_m^d = \left[B_{i.m}^d\right]_{Nx1}, \tag{A.8}$$

and write equation (A.5) in matrix form to obtain

$$\mathcal{P}_m = \mathcal{L}_m + R_m^{(2)} \beta_m \mathcal{P}_m. \tag{A.9}$$

Since the solution to (A.9) is unique, this establishes that the matrix $\left(I - R_m^{(2)}\beta_m\right)$ has an inverse.

Rewrite equation (A.4) in matrix form as follows.

$$\mathcal{P}_m = \mathcal{L}_m + \mathcal{L}_m \left[R_m^{(1)} - R_m^{(2)}\right] \beta_m + R_m^{(2)} \beta_m \mathcal{P}_m. \tag{A.10}$$

Since $\left(I - R_m^{(2)}\beta_m\right)$ has an inverse, we can conclude that for the case of two levels of mobility $\mathcal{P}_m$ is also unique.

A.2 Forced Termination Probability

Let $P_{j,k}^{(m)}$ be the probability that a call originated in cell j of class m is forced terminated at the k^{th} handoff. Taking into account the two levels of mobility we can see, for example, that

$$P_{j,3}^{(m)} = \sum_{i \in \mathcal{A}_j} q_{ji,m}^{(1)} \left(1 - B_{hi,m}\right) \sum_{k \in \mathcal{A}_i} q_{ik,m}^{(2)} \left(1 - B_{hk,m}\right) \sum_{z \in \mathcal{A}_k} q_{kz,m}^{(2)} B_{hz,m}. \tag{A.11}$$

Therefore, by using the matrices defined above, we can express (A.11) in matrix form defining $P_3^{(m)} = \left[P_{j,3}^{(m)}\right]_{N \times 1}$ as follows

$$P_3^{(m)} = R_m^{(1)} \beta_m R_m^{(2)} \beta_m R_m^{(2)} B_{hm}, \tag{A.12}$$

and generalizing for any $k \geq 1$ we get

$$P_{k+1}^{(m)} = R_m^{(1)} \beta_m (R_m^{(2)} \beta_m)^{k-1} R_m^{(2)} B_{hm}. \tag{A.13}$$

Hence the forced termination probability will be given by

$$\begin{aligned}
B_m^d &= \sum_{k=2}^{\infty} P_k^{(m)} + R_m^{(1)} B_{hm} \\
&= R_m^{(1)} \beta_m \sum_{k=2}^{\infty} (R_m^{(2)} \beta_m)^{k-2} R_m^{(2)} B_{hm} + R_m^{(1)} B_{hm} \\
&= R_m^{(1)} \beta_m \sum_{k=0}^{\infty} (R_m^{(2)} \beta_m)^{k} R_m^{(2)} B_{hm} + R_m^{(1)} B_{hm} \\
&= R_m^{(1)} \beta_m \left[I - R_m^{(2)} \beta_m\right]^{-1} R_m^{(2)} B_{hm} + R_m^{(1)} B_{hm}.
\end{aligned} \tag{A.14}$$

The last equality follows by the uniqueness argument above, where it was shown that the inverse of that matrix exists.

A.3 Net Revenue Derivation

In this section we show that (7.9) can be expressed as (7.10). Throughout this derivation, the dependence of the variables on the different vectors defined in Section 7.3 is not shown. In order to obtain this we have to see that for each class m we have the following equality

$$\sum_{j \in \mathcal{N}} \lambda_{j,m} (1 - B_{j,m}) c_{j,m} B_{j,m}^d = \sum_{j \in \mathcal{N}} c_{j,m} B_{hj,m} \cdot \sum_{k \in \mathcal{A}_j} \nu_{kj,m}. \tag{A.15}$$

We can express the Left Hand Side (LHS) of (A.15) in matrix form using the definitions in (A.6), (A.7) and (A.8) as follows

$$
\begin{aligned}
\text{LHS} &= \mathbf{c}_m \mathcal{L}_m \gamma_m B^d \\
&= \mathbf{c}_m \mathcal{L}_m \gamma_m \left\{ R_m^{(1)} \beta_m \left[I - R_m^{(2)} \beta_m \right]^{-1} R_m^{(2)} B_{hm} + R_m^{(1)} B_{hm} \right\} \\
&= \mathbf{c}_m \mathcal{L}_m \gamma_m R_m^{(1)} \left\{ I + \beta_m \left[I - R_m^{(2)} \beta_m \right]^{-1} R_m^{(2)} \right\} B_{hm}, \qquad \text{(A.16)}
\end{aligned}
$$

and using the identity $I + D(I - ED)^{-1}E = (I - DE)^{-1}$ in (A.16) we get

$$
\text{LHS} = \mathbf{c}_m \mathcal{L}_m \gamma_m R_m^{(1)} \left(I - R_m^{(2)} \beta_m \right)^{-1} B_{hm}. \qquad \text{(A.17)}
$$

For the Right Hand Side (RHS) of (A.15) we first consider the handoff rates $\nu_{kj,m}$ given by (7.5) and sum over k giving the total handoff rate into cell j, then we have

$$
\sum_{k \in \mathcal{A}_j} \nu_{kj,m} = \sum_{k \in \mathcal{A}_j} \lambda_{k,m} (1 - B_{k,m}) q_{kj,m}^{(1)} + \sum_{k \in \mathcal{A}_j} (1 - B_{hk,m}) q_{kj,m}^{(2)} \sum_{x \in \mathcal{A}_k} \nu_{xk,m}, \qquad \text{(A.18)}
$$

and define $x_j = \sum_{k \in \mathcal{A}_j} \nu_{kj,m}$, then (A.18) can be rewritten as

$$
x_j = \sum_{k \in \mathcal{A}_j} \lambda_{k,m} (1 - B_{k,m}) q_{kj,m}^{(1)} + \sum_{k \in \mathcal{A}_j} (1 - B_{hk,m}) q_{kj,m}^{(2)} x_k, \qquad \text{(A.19)}
$$

note that the RHS of (A.15) is given by $\sum_{j \in \mathcal{N}} c_{j,m} B_{hj,m} \cdot x_j$ for which we can obtain the product $c_{j,m} x_j$ using (A.19). Then, we obtain in matrix form with the definitions (A.6), (A.7) and (A.8)

$$
\mathbf{c}_m X = \mathbf{c}_m \mathcal{L}_m \gamma_m R_m^{(1)} \left(I - R_m^{(2)} \beta_m \right)^{-1}, \qquad \text{(A.20)}
$$

where X is a diagonal matrix with x_j in the j^{th} element of the diagonal and which substituted in the RHS of (A.15) gives the desired result

$$
\text{RHS} = \mathbf{c}_m \mathcal{L}_m \gamma_m R_m^{(1)} \left(I - R_m^{(2)} \beta_m \right)^{-1} B_{hm}. \qquad \text{(A.21)}
$$

Appendix B. Notation

(j, k) : Link whose origin is node j and destination is node k. (Chapters 3, 4, 5 and 6.)

$[j, k]$: OD pair whose direct link is (j, k). (Chapters 3, 4, 5 and 6.)

A_{jk} : Set of alternate routes for OD pair $[j, k]$. (Chapters 3, 4, 5 and 6.)

$\mathcal{A}_j$: Set of adjacent cells to cell j. (Chapter 7.)

$\mathcal{A}_n^{ij}$: mth aggregate state of link (i, j). (Chapters 3, 4, 5 and 6.)

$\alpha_{jk}(m)$: Total offered traffic to link (j, k), when it is in state m. (Chapters 3, 4 and 5.)

$\alpha_{jk}^{(s)}(\mathbf{n})$: Total offered traffic of class s to link (j, k), when it is in state $\mathbf{n}$. (Chapter 6.)

α_j : Total offered traffic to cell j, when in the reserved states. (Chapter 7.)

$\alpha_{j,s}$: Total offered traffic of class s to cell j, when it is in the reserved states. (Chapter 7.)

$\mathcal{B}_i^{(s)}$: Set of blocked states of class s in cell i. (Chapter 7.)

$\mathcal{B}_{ij}^{(s)}$: Set of blocked states of class s in link (i, j). (Chapter 6.)

B_{jk} : Blocking probability for OD pair $[j, k]$. (Chapters 3, 4 and 5.)

$B_{jk}^{(s)}$: Class s blocking probability for OD pair $[j, k]$. (Chapter 6.)

B_s^d : Forced termination probability of class s traffic for cell j. (Chapter 7.)

B_j : New call blocking probability for cell j. (Chapter 7.)

$B_{j,s}$: Class s New call blocking probability for cell j. (Chapter 7.)

B_{hj} : Handoff call blocking probability for cell j. (Chapter 7.)

$B_{hj,s}$: Class s handoff call blocking probability for cell j. (Chapter 7.)

$\beta_{ij}(k)$: Load sharing coefficient of OD pair $[i,j]$ for route that uses link (i,k). (Chapters 3, 4 and 5.)

b_m : Bandwidth requirement for traffic class m. (Chapters 6 and 7.)

C_{jk} : Capacity of link (j,k). (Chapters 3, 4, 5 and 6.)

C_j : Capacity of cell j. (Chapter 7.)

$C_{j,1}$: Capacity of cell j. (Chapter 7.)

$C_{j,2}$: Reservation parameter cell j. (Chapter 7.)

c_j : Cost generated by rejection of a handoff call on cell j. (Chapter 7.)

$c_{j,s}$: Cost generated by rejecting a handoff call of class s on cell j. (Chapter 7.)

K : Number of aggregate states (Chapters 3, 4 and 5) and total number of traffic classes (Chapter 6.)

$\mathcal{L}$: Set of links. (Chapters 3, 4, 5 and 6.)

L : Network blocking probability. (Chapters 3, 4 and 5.)

λ_{jk} : External arrival rate for OD pair $[j,k]$. (Chapters 3, 4, 5 and 6.)

$\lambda_{jk}^{(s)}$: External arrival rate for traffic class s for OD pair $[j,k]$. (Chapter 6.)

λ_j : New call arrival rate for cell j. (Chapter 7.)

$\lambda_{j,s}$: New call arrival rate of class s for cell j. (Chapter 7.)

$\mathcal{N}$: Set of Nodes. (Chapters 3, 4, 5 and 6.) Set of cells. (Chapter 7.)

N : Number of nodes (Chapter 3, 4, 5 and 6) and of cells (Chapter 7.)

M : Total number of traffic classes. (Chapter 7.)

μ_i : Service rate in cell i. (Chapter 7.)

$\mu_{i,s}$: Service rate for traffic class s in cell i. (Chapter 7.)

Ω_{jk} : Set of feasible states for link (Chapters 3, 4, 5, 6 and 7.).

$\mathcal{O}$: Set of Origin-Destination pairs (O-D pairs). (Chapters 3, 4, 5 and 6.)

$p_{jk}(m)$: Probability distribution for the states of link (j, k). (Chapters 3, to 5.)

$p_{jk}(\mathbf{n})$: Probability distribution for the states of link (j, k). (Chapter 6.)

$p_j(m)$: Probability distribution for the states of cell j. (Chapter 7.)

$p_j(\mathbf{n})$: Probability distribution for the states of cell j. (Chapter 7.)

$\mathcal{Q}_i^{(s)}$: Set of reserved states of class s in cell i. (Chapter 7.)

$\mathcal{Q}_{ij}^{(s)}$: Set of reserved states of class s in link (i, j). (Chapter 6.)

$q_{ij}^{(1)}$: Probability that a new call from cell i handoffs to cell j. (Chapter 7.)

$q_{ij}^{(2)}$: Probability that a handoff call from cell i handoffs to cell j. (Chapter 7.)

$q_{ij,s}^{(1)}$: Probability that a new call of class s from cell i handoffs to cell j. (Chapter 7.)

$q_{ij,s}^{(2)}$: Probability that a handoff call of class s from cell i handoffs to cell j. (Chapter 7.)

$\mathcal{R}$: Set of routes in the network. (Chapters 3, 4, 5 and 6.)

ρ_j : Total offered traffic to cell j, when it is in the unreserved states. (Chapter 7.)

$\rho_{j,s}$: Total offered traffic of class s to cell j, when it is in the unreserved states. (Chapter 7.)

r_n^{ij} : nth Threshold value for the nth aggregate state of link (i,j). (Chapters 3, 4, 5 and 6.)

S_{jk} : Set of links adjacent to link (j,k). (Chapters 3, 4 and 5.)

T : Trunk reservation parameter. (Chapters 3, 4, 5 and 6.)

$\mathcal{T}_{ij}$: Set of intermediate nodes for OD pair $[i,j]$. (Chapters 3, 4, 5 and 6.)

T_j : Channel reservation parameter cell j. (Chapter 7.)

T_s : Channel reservation parameter for class s in cell j. (Chapter 7.)

$\mathcal{U}_i^{(s)}$: Set of unreserved states of class s in cell i. (Chapter 7.)

$\mathcal{U}_{ij}^{(s)}$: Set of unreserved states of class s in link (i,j). (Chapter 6.)

ν_{ij} : Handoff rate from cell i to cell j. (Chapter 7.)

$\nu_{ij,s}$: Handoff rate of traffic class s from cell i to cell j. (Chapter 7.)

$\nu_{ib,j}$: Contribution of OD pair $[i,b]$ to the offered traffic of link (i,j). (Chapters 3 and 4.)

W : Network rate of return. (Chapters 4, 5, 6 and 7.)

w_{ij} : Revenue generated by accepting a call on OD pair $[i,j]$. (Chapters 4 and 5.)

w_j : Revenue generated by accepting a new call on cell j. (Chapter 7.)

$w_{j,s}$: Revenue generated by accepting a new call of class s on cell j. (Chapter 7.)

Vita

Cesar Vargas Rosales was born in México city on December 23rd, 1963. He obtained the Bachelor's degree in Mechanical and Electrical Engineering from the Universidad Nacional Autónoma de México (UNAM) in 1988, with a thesis about Communications and Control titled "Automated Control of Electrical Substations through the Altern Current Line." In December 1992, he obtained the Master's degree in Electrical Engineering majoring in Communications and Signal Processing from Louisiana State University, and currently he is a candidate for the Doctorate degree in Electrical Engineering majoring in Communications and Signal Processing from Louisiana State University with a minor in Mathematics and a peripheral area in Control Systems. His thesis research has been concentrated in Adaptive Routing schemes analysis, optimization and performance of networks using shadow prices.

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Title of Dissertation: Communication Network Design and Evaluation Using Shadow Prices

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Date of Examination:

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